MATE 6727 assignment 1

1. Assume that X is infinite, and that there exists a surjection $\phi : N \to X$. Define recursively the sequence (n_p) by setting $n_0 = 0$ and

$$n_{p+1} = \min\left\{ n : \phi(n) \notin \{ \phi(n_0), \phi(n_1), \ldots, \phi(n_p) \} \right\} \text{ for } p \in N$$

   Show that this sequence is well-defined, and that the map $p \to \phi(n_p)$ is a bijection from N to X.

2. Which, if any, of the following sets is countable?

   (a) The set of sequences of integers.

   (b) The set of sequences of integers that are zero after a certain place.

   (c) The set of sequences of integers that are constant after a certain place.

3. Let A be an infinite set and B be a countable set. Prove that there is a bijection between A and $A \cup B$.

4. Let X be a connected metric space that contains at least two points. Prove that there exists an injection from [0,1] into X. Deduce that X is not countable.
   Hint. Let x and y be distinct points of X. Prove that for every $r \in [0, d(x, y)]$, the set

$$S_r = \{ t \in X : d(x, t) = r \}$$

   is nonempty.

5. Let f be an increasing function from I to R, where I is an open, nonempty interval of R. Let S be the set of discontinuity points of f. If $x \in I$, denote by $f(x_+)$ and $f(x\_)$ the right and left limit of f at x (they exist, since x is monotone).

   (a) Prove that $S = \{ x \in I : f(x\_) < f(x_+) \}$.

   (b) For $x \in S$, write $I_x = \big(f(x\_), f(x_+)\big)$. By considering the family $(I_x)_{x \in S}$, prove that S is countable.

   (c) Conversely, let $S = \{x_n\}_{n \in N}$ be a countable subset of I. Prove that there exists an increasing function the set of points of discontinuity of which is exactly S.
   Hint. Put $f(x) = \sum_{n=0}^{+\infty} 2^{-n} 1_{[x_n, +\infty)}(x)$.