MATE 6627 assignment 2

6 Show that if a normed space E contains a countable fundamental family of vectors $\{x_0, x_1, \ldots\}$, it is separable.
Hint. Show that a certain family of linear combinations of the set $\{x_0, x_1, \ldots\}$ spans a dense set in E.

7 Let X be a metric space. We say that a family of open sets $(U_i)_{i \in I}$ of X is a *basis of open sets* (or *open basis*) of X if, for every nonempty open subset U of X and for every x in U, there exists $i \in I$ such that $x \in U_i \subset U$.

a) Let $\mathcal{U}$ be an open basis of X. Prove that any open set U in X is the union of the elements of $\mathcal{U}$ contained in U.

b) Prove that X is separable just in case it has a countable open basis.
Hint. If (x_n) is a dense subsequence in X, the family

$$\big(B(x_n, 1/(p+1))\big)_{n,p \in N}$$

is an open basis of X. Conversely, if (U_n) is an open basis of X, any sequence (x_n) with the property that $x_n \in U_n$ for every n is dense in X.

8 Let X be a separable metric space.

a) Let $f : X \to R$ be a function, and let M be the set of points of X where f has a local extremum. Prove that f(M) is countable.
Hint. Let M^+ be the set of points of X where f has a local maximum and let $\mathcal{U}$ be a countable open basis of X. Prove that there is an injection from $f(M^+)$ into $\mathcal{U}$.

b) Prove that a continuous function $f : R \to R$ that has a local extremum at every point is constant.

9 If x and y are real numbers, we write $d(x, y) = |y - x|$ and $\delta(x, y) = |\arctan y - \arctan x|$. Prove that δ is a metric on R equivalent to the usual metric d; that is, the two metrics define the same open sets. Show that (R, δ) is precompact, but (R, d) is not.

10 Prove that a metric space X is precompact just in case every sequence in X has a Cauchy subsequence.

11 Let A be a subset of a normed vector space E. Prove that A is precompact if and only if A is bounded and, for every $\varepsilon > 0$, there exists a finite-dimensional vector subspace F_ε of E such that $d(x, F_\varepsilon) \le \varepsilon$ for all x in A.