MATE 6677 assignment 1

1. State, in generality, the inverse function theorem.

2. State, in generality, the implicit function theorem. Illustrate it on the two points $(0, 1, 0)$ and $(-1, 0, 0)$ of the unit sphere of R^3.

3. Using a suitable change of variable in the integral, and the formula $\int e^{-\pi x^2}\ dx = 1$, show that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\,.$$

4. Using Hölder's inequality, show that, if $1 < p < q$, and if Ω has finite volume, then $L^q(\Omega) \subset L^p(\Omega)$.

5. Show that if V is a compact set contained in an open set D, then, for some $\delta > 0$,

$$V + \delta B(0, 1) \subset D.$$

   That is to say, any point at distance less than δ from some point of V must belong to D. Use directly the definition of compactness, and cover V by some balls.

6. Let Ω be an open domain. Show, using the mean-value theorem, that $C^1(\Omega) \subset C^\alpha(\Omega)$ if $0 < \alpha < 1$. (Use the result of the previous problem to insure that $u(x + y)$ is defined for "y sufficiently close to 0" in the definition of the space $C^\alpha(\Omega)$).

Marks: $6 + 6 + 6 + 9 + 9 + 6 = 42$.