MATE 6677 assignment 6

25. Compute the integral of $N(x)$ on the unit ball in dimensions $n = 2$ and $n > 2$.

26. Show that if $|x| \leq R_0$, then, given ε, for R large enough,

$$|y| \geq R \Rightarrow 1 - \varepsilon \leq \frac{|y - x|}{|y|} \leq 1 + \varepsilon.$$

27. Show that the sequence u_j defined in the proof of (2.16) tends to u as distributions as $j \to \infty$.

28. Let ϕ be a test function. Show that, for h in a certain neighbourhood of 0 and $M > 0$ a certain constant,

$$|\phi(x + he_j) - \phi(x) - h\partial_j \phi(x)| \leq Mh^2$$

(uniformly in x).

29. Let u be a distribution of compact support, and $\psi \in \mathcal{D}$. Show that $u * \psi \in \mathcal{D}$. These are the steps:

   (a) $u * \psi(x) = \langle u, (\tilde{\psi})_{-x} \rangle$ is a continuous function of x (see problem 17). The way to prove this is to show that $x \to (\tilde{\psi})_{-x}$ is a continuous function $R \to \mathcal{D}$.

   (b) The support of $u * \psi(x)$ is compact.

   (c) $u * \psi(x)$ is infinitely differentiable. For this, it is enough to show that, for all multi-indices β,

$$\partial^{\beta}(u * \psi) = u * \partial^{\beta}(\psi) \qquad (1).$$

   In turn, it is enough to show that, for $1 \leq j \leq n$,

$$\partial_j(u * \psi) = u * \partial_j \psi \qquad (2)$$

   (why?) To show (2), do not use the result in Folland (which we did not completely prove), but do it directly, using the definition of derivative as limit of a difference quotient: for each x,

$$\lim_{h \to 0} \frac{1}{h} \big(u * \psi(x + he_j) - u * \psi(x) \big) = u * \partial_j \psi(x)$$

   This, in turn, will be true if for each x,

$$\lim_{h \to 0} \frac{1}{h} \big(\tilde{\psi}_{-(x+he_j)} - \tilde{\psi}_{-x} \big) = (\partial_j \psi\tilde{)}_{-x}$$

(limit in $\mathcal{D}$).

(a) and (b), if worked out, should be part of the re-work for problems 17 and 18. So your work for this problem will consist of part (c).

Marks: 4 + 6 + 6 + 6 + 10 = 32