MATE 6677 assignment 7

30. For certain domains, the Green's function may be computed explicitly. Take Ω to be the unit disc in the plane. Set

$$v(x,y) = G(x,y) - \frac{1}{2\pi}\log|x-y|$$

where $v(x,\ \cdot\ )$ is required to be harmonic in Ω and $G(x,\ \cdot\ )$ vanishes on the boundary Γ of the unit disc. The idea of images is to express v as the potential due to sources lying outside Ω. Let y^* be the inverse point of y with respect to the unit circle (i.e., $y^* = y/|y|^2$.) Show that

$$G(x,y) = \frac{1}{2\pi}\log|x-y| - \frac{1}{2\pi}\log|x-y^*| - \frac{1}{2\pi}\log|y|.$$

In particular, if $x \neq 0$, show that $G(x,\ \cdot\ )$ can be extended by continuity to $y = 0$. What is the normal derivative of G on the unit circle?

31. Show that the heat kernel satisfies the inequality

$$K(t,x,y) \leq \pi^{-\frac{n}{2}}|x-y|^{-n}\left(\frac{n}{2e}\right)^{\frac{n}{2}}$$

for $x, y \in R^n$, $t > 0$.

32. Let $u(t,x)$ be a function of class C^1, and

$$B_t = \{x : |x| \leq t\}, \qquad t > 0.$$

Show that if

$$E(t) = \int_{B_t} u(t,x)\,dx,$$

then

$$\frac{dE(t)}{dt} = \int_{B_t} \partial_t u\,dx + \int_{\partial B_t} u\,d\sigma(x).$$

Marks: 16 + 6 + 9 = 31