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Students' understanding of quadratic equations

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Action–Process–Object–Schema theory (APOS) was applied to study student understanding of quadratic equations in one variable. This required proposing a detailed conjecture (called a genetic decomposition) of mental constructions students may do to understand quadratic equations. The genetic decomposition which was proposed can contribute to help students achieve an understanding of quadratic equations with improved interrelation of ideas and more flexible application of solution methods. Semi-structured interviews with eight beginning undergraduate students explored which of the mental constructions conjectured in the genetic decomposition students could do, and which they had difficulty doing. Two of the mental constructions that form part of the genetic decomposition are highlighted and corresponding further data were obtained from the written work of 121 undergraduate science and engineering students taking a multivariable calculus course. The results suggest the importance of explicitly considering these two highlighted mental constructions.

Keywords: quadratic equations; APOS theory; genetic decomposition

1. Introduction

The understanding of quadratic equations with one unknown is fundamental for advanced studies in mathematics and other sciences. Nevertheless, it has been found in various investigations that many secondary school students and even undergraduate students do not truly understand these equations or the rules they use to solve them. For example, as Didis, Baş, and Erbaş [1] concluded: 'Although students knew some rules related to solving quadratics, they applied these rules thinking about neither why they did so, nor whether what they were doing was mathematically correct. It was concluded that the students' understanding in solving quadratic equations is instrumental (or procedural), rather than relational (or conceptual).' Hence it is important to study how students learn to solve quadratic equations so that instruction in this topic may be improved.

Different explanations for the scarce understanding of quadratic equations have been suggested. Sönnerhead,[2] for example, noticed that mathematics textbooks in Sweden omit important concepts that would not be presented by many teachers, thus students will tend to develop a disconnected and incomplete set of ideas regarding quadratic equations. Lima and Tall [3] interviewed students who were taught to solve quadratic equations with a strong emphasis on using the quadratic formula as a general solution method for any type of equation. They conclude: 'such a strategy enabled a small number of students to be able to solve specific quadratic equations, but it did not help in general to encourage

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students to construct flexible meanings in algebra'. A flexible understanding would allow adjusting the solution method to the type of quadratic equation and would require, as recommended by Kotsopoulos,[4] using different types of quadratic equations, not only on standard form $ax^2 + bx + c = 0$, but also on factored form $a(x - r_1)(x - r_2) = 0$, and vertex form $a(x - h)^2 + k = 0$. Moreover, Olteanu and Holmqvist,[5] when comparing differences in student learning of the quadratic formula as a result of differences in their teaching, observed that a more successful teacher gave students different opportunities to experience variations in the form of the quadratic equation, and to discern the way in which the parts of equations are related to each other. Further support for this idea is provided by Eraslan [6] when in his discussion of quadratic functions he argued that students have difficulty relating the vertex form of a quadratic equation to its standard form, preferring the latter form. On the other hand, Bosse and Nandakumar [7] observed that the probability that a given quadratic with integer coefficients in the interval $[-10, 10]$ has rational roots is only 15% and that this gets smaller as the interval of possible coefficients increases. Hence, it is unlikely that a quadratic equation resulting from an application of science or used to solve a real-world problem can be solved using factoring techniques. They argue that this is reason enough to emphasize the other techniques of square roots, completing the square, and the quadratic formula. Furthermore, Gray and Thomas [8] reported on an experiment where students, who had received lessons using the graphing calculator, showed difficulty relating processes for the graphical and symbolic solution of quadratic equations.

Some investigations on student understanding of quadratic equations refer to specific misconceptions. Vaiyavutjamai, Ellerton, and Clements,[9] Bosse and Nandakumar,[7] and Ochoviet and Oktaç [10,11] observed, for example, that some students believe that the variable in an equation of the form $(x + a)(x + b) = c$ may have different values at once, which means that many students are not aware of the relation of the solution with the original quadratic equation. Also, as cited in the Vaiyavutjamai et al.,[9] many of the students did not realize that quadratic equations often had two solutions. In addition, Vaiyavutjamai et al. [9] and also, Tall, Lima, and Healy [12] found that most of the students in their study could not find the correct solutions of the equation $x^2 = 9$ by using correct procedures or correct explanations. Most of the students either they only found one solution or assumed that $\sqrt{9} = \pm 3$. In this and other examples of common errors and misconceptions they considered, Tall et al. [12] argued that students were for the most part shifting symbols around in a procedural embodied sense rather than using the more general reasoning of 'doing the same on both sides'. Also, Didis et al. [1] observed that students were not aware of the missing root 0 when cancelling an x in the equation $2x^2 = 3x$, and generally did not show a good understanding of the zero product theorem, a fact also observed by Ochoviet and Oktak [10,11] and Bosse and Nandakumar.[7]

All of these investigations have their own context, with many of them involving observations made with pre-university students. In Puerto Rico, an unincorporated territory of the United States, the Department of Education establishes that the 8th grade students will learn how to solve simple quadratic equations by factorization and the zero product property, but it is not until 10th grade that students learn how to solve quadratic equations, not only simple quadratic equation, by the following techniques: factoring, the square root method, completing the square, the quadratic formula, and using technology. Hence, Puerto Rican students are expected to have seen quadratic equations in two different stages in their respective schools before beginning university studies. Given the context of the Puerto

Rican beginning university student, this study proposes to investigate student understanding of quadratic equations by

(1) establishing a conjecture of the mental constructions (stated in terms of the constructs of Action–Process–Object–Schema (APOS) theory, as will be discussed further along) that beginning university students may do in order to understand how to solve quadratic equations;
(2) using semi-structured interviews in order to investigate which of the conjectured mental constructions students can do and which they have difficulty doing; and
(3) using written work from more advanced undergraduate students to investigate their use and understanding of two specific mental constructions conjectured in the genetic decomposition.

2. Theoretical framework

APOS theory will be used as theoretical framework to study the level of cognitive development of students who completed a precalculus course using a traditional lecture/recitation model, as discussed in Arnon et al.[[13], p.106] APOS theory was chosen since it has been used to study student learning of a variety of different mathematical concepts and has proven to give important insights on students learning of mathematics (Arnon et al. [13] has an annotated bibliography). Also, it has been tested in the classroom and has proven effective in promoting students' learning of different concepts and guiding the development of classroom activities.

In APOS theory (for more details, see [13]), an *Action* is a transformation of a mathematical object performed by an individual that the individual perceives as external. It may be a transformation where the individual is limited to following an explicit algorithm step by step or is limited to the rigid application of a memorized fact. An individual who is limited to performing actions when dealing with a problem situation that involves a particular mathematical notion is said to have an action conception of the mathematical notion. So, for example, a student who needs to be given the quadratic formula or who has memorized the quadratic formula and is only able to think of using it when a quadratic equation is given in standard form, or who is unable to anticipate or discuss the nature of its solutions without explicitly computing the solutions would show behaviour consistent with an action conception of the quadratic formula.

If the individual repeats an action and reflects on it, the action may be interiorized into a *Process*. The process is now perceived as internal, under control of the individual. An individual with a process conception of a mathematical notion may reflect on it without having to explicitly carry out all the steps of the transformation. A process may be reversed and it may be coordinated with other processes. For example, a student who can anticipate being able to use the quadratic formula to find solutions of a quadratic equation regardless of the form in which the quadratic equation is given, or who without prompting can use the discriminant to discuss the nature of the solutions, or who can relate the nature of the solutions to the graphical representation of the corresponding quadratic function, would be showing behaviour consistent with at least a process conception of the quadratic formula.

As an individual needs to perform actions on a process, he/she may become aware of the process as a totality, an entity in itself. When the individual can perform or imagine performing actions on the process, it is said the process has been encapsulated into an *Object*. An object may be de-encapsulated into the process and actions it came from as needed in a problem situation. For example, a student having an object conception of quadratic formula can be expected to be able to use it if necessary, without prompt, and in any context, be it in a different course, with 'hidden' quadratics, or with quadratics whose coefficients are variable expressions.

A mathematical *Schema* is a coherent collection of actions, processes, objects, and other previously constructed schemas, which are synthesized to form mathematical structures utilized in problem situations.[14] These schemas evolve as new relations between new and previous action, process, and object conceptions, and other schemas are constructed and reconstructed. Their evolution may be described by three stages that Piaget and García [15] refer to as the 'triad': At the general *intra*-stage some operational actions are possible, but there is an absence of relationships between properties. At the *inter*-stage, the identification of relations between different processes and objects, and transformations are starting to form, but they remain isolated. The *trans*-stage is defined in terms of the construction of a synthesis between them to form a coherent structure.[16] For example, in the genetic decomposition that we are about to describe, different processes and objects for solving quadratic equations using square roots, completion of square, quadratic formula, factoring, and graphical interpretation are given. The stage of development (intra-, inter-, trans-) of the schema of quadratic equations is a measure of the degree of interconnectedness of these ideas in the students' minds.

The progression from action, to process, to object, and to having such constructions organized in schemas is a dialectical progression where there may be passages and returns from one type of construction to the other.[17] What the theory states is that a student's tendency to deal with problem situations in diverse mathematical tasks involving a particular mathematical concept is different depending on whether the student understands the concept as an Action, a Process, or an Object or has constructed a coherent Schema. Hence an individual's mental construction of a particular mathematical concept may be classified (as action, process, object, intra-schema, inter-schema, or trans-schema) by inference made from observations of his/her overall behaviour when using or applying the mathematical concept in a diverse group of problem situations.

In APOS theory, research starts with a conjecture of the mental constructions (in terms of the constructs of the theory) that students may do in order to understand a particular mathematical concept. The conjecture, called a genetic decomposition, is based on the mathematical concept itself, on the classroom experience of the researchers, and results from any available data. The conjecture is then tested by doing student interviews. What typically happens is that students will give evidence of doing some unexpected mental constructions and will also show difficulty on some of the conjectured constructions. This leads to refining the genetic decomposition to better reflect the mental constructions that students actually do and it also leads to the design of student activities and more effective pedagogies to help them make particular constructions where they have shown difficulty. This marks the end of a research cycle and the beginning of the next one (see Figure 1) which would start with the class testing of the specially designed activities. Iterations of this cycle of research continue until stability is reached, that is, a genetic decomposition is obtained that serves to predict the mental constructions that students can actually do to understand the mathematical concept and also serves as a guide for the instruction of the particular mathematical concept. Our study is the first cycle of an APOS-based research project dealing with student understanding of quadratic functions. The design of didactic material based on the refined genetic decomposition and its classroom implementation is not discussed in this article.

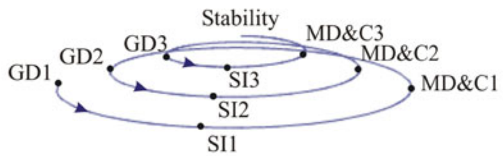


Figure 1. APOS research cycles.
Note: GD – genetic decomposition; SI – student interviews; MD&C – didactic materials development and class testing

3. Initial genetic decomposition

The researchers' experience is one of the components informing a genetic decomposition. In our case, one of the researchers has over 30 years of experience teaching which includes writing two precalculus textbooks used for many years at his university and observing for many years how students at a multivariable calculus course struggle with some precalculus ideas. While a genetic decomposition need not be unique, perhaps researchers with a different experience would have presented a different genetic decomposition, what is important is that the data obtained from students support the decomposition. As suggested by Figure 1, a genetic decomposition can be expected to develop with subsequent implementation of class activities (to help students make mental constructions where they have shown difficulty) and further student interviews (to verify that with the implemented classroom activities students are able to make the conjectured mental constructions and/or to obtain more in-depth information about some aspects of the genetic decomposition). The main components of the schema for quadratic equations given in our genetic decomposition describe mental constructions leading to four sub-schemas, each corresponding to a solution method: square root (includes completing the square), quadratic formula, factoring, and graphical method. The present study does not explore actions and processes needed to interrelate the different sub-schemas and thus obtain the trans-stage of development required for a coherent schema. This is left for further studies. We should stress that whenever the genetic decomposition describes the mental construction of a process, this presupposes that students will be engaged in activities designed to help them make the desired process mental construction. So, the mere mention of a property on the blackboard or text does not substitute the repetition and reflection on actions needed for interiorization into a process.

3.1. *Prerequisite*

Clearly, there are many pre-requisites necessary for student understanding of quadratic equations, as, for example, students should have a process conception for solving linear equations and a schema of properties of real numbers. We do not intend to make a complete list. However, we do single out that students should have a process to eliminate absolute value. This may be constructed by interiorizing the action of solving equations of the form $|x - a| = b$, geometrically, numerically, and symbolically into a process that recognizes that if $b \geq 0$ and $|y| = b$ then $y = \pm b$.

3.2. *Sub-schema for solving quadratic equations using the square root (SR)*

SR1 The action of numerically exploring the possible solutions of the equations $\sqrt{x^2} = x$ and $\sqrt{x^2} = |x|$ followed by the action of numerically exploring the solutions to equations of the form $\sqrt{(f(x))^2} = f(x)$, and $\sqrt{(f(x))^2} = |f(x)|$ may be interiorized into a process of recognition of the property $\sqrt{(f(x))^2} = |f(x)|$. The need to perform actions on a process can help its encapsulation into an object; hence, the action of reducing the exponent in equations of the form $(f(x))^2 = a$ by taking the square root on both sides to obtain $\sqrt{(f(x))^2} = \sqrt{a}$, and from here conclude that$|f(x)| = \sqrt{a}$ may enable students to attain an object conception of the before-mentioned property.

SR2 This last process (SR1) is coordinated with the process for solving absolute value equations ($|y| = b \Rightarrow y = \pm b$) to obtain a process that recognizes that if $b \geq 0$ and $x^2 = b$, then $x = \pm\sqrt{b}$.

SR3 The need to perform actions on this last process to solve equations of the form $(ax + b)^2 = c$, where $a \neq 0$, $c \geq 0$ may lead to encapsulating the process into an object conception of solving a perfect square using square root.

SR4 The action of adding $(b/2)^2$ to expressions of the form $x^2 + bx$ to obtain perfect squares $(x + b/2)^2$ is interiorized into the process of completing the square that recognizes that a quadratic expression with leading coefficient 1 can be changed into a perfect square by adding a quantity.

SR5 The process in SR4 is coordinated with processes related to basic properties of the real numbers (needed to maintain equality) and with the process to solve a perfect square (see SR3) in order to solve quadratic equations *equivalent* to the form $x^2 + bx + c = 0$ (thus the equations may not be in standard form). This results in a process for solving monic quadratic equations (with leading coefficient 1) by completing the square.

SR6 The action of reducing a quadratic equation equivalent to the form $ax^2 + bx + c = 0$ (observe that now the leading coefficient need not be (1) a monic equation and then solving the resulting equation by completing the square is an action on the process in SR5. This allows the encapsulation of the process in SR5 into an object conception of solving quadratic equations by completing the square.

3.3. *Sub-schema for solving quadratic equations using the quadratic formula (QF)*

QF1 Repeating and reflecting on the action of solving symbolically the equations $x^2 + bx + c = 0$ (by completing the square where b and c are left as unknowns) and $ax^2 + bx + c = 0$, and performing actions that explore numerically the type of solutions that can be obtained by the quadratic formula are interiorized into a process of recognition that the quadratic formula produces all solutions of a quadratic equation and that there can be 0, 1, or 2 solutions.

QF2 The action of applying the quadratic formula to solve equations *equivalent* to $ax^2 + bx + c = 0$ is interiorized into a process to solve quadratic equations using the quadratic formula.

QF3 Apply the action of substituting $u = f(x)$ into equations *equivalent* to $a(f(x))^2 + b(f(x)) + c = 0$ (where $f(x) \neq mx$ for all real numbers m; these equations will henceforth be called 'hidden quadratics') and use the quadratic formula to solve the resulting equation $au^2 + bu + c = 0$ (of course, any solution $u = s$ so obtained will require coordinating with other processes in order to obtain the solutions of the original equation from $f(x) = s$). Different types of functions $f(x)$ should be used in the substitution. This is an action on the process (QF2) and hence enables encapsulating the process into an object conception of the quadratic formula.

3.4. *Sub-schema for solving quadratic equations by factoring (F)*

F1 Coordinate a process of factoring with a process conception of the null product theorem to solve equations *equivalent* to $x^2 + (a + b)x + ab = 0$, where a and b are integers. This results in a process to solve quadratic equations with leading coefficient 1 by factoring.

F2 Coordinate a process of factoring with the process of the null product theorem to solve equations *equivalent* to $(ax + b)(cx + d) = 0$, where a, b, c, and d are integers. This results in a process to solve quadratic equations by factoring.

3.5. *Sub-schema for solving quadratic equations graphically (G)*

G1 The action of finding the x coordinates of the points of intersection of the graphs of $y = ax^2 + bx + c$ and $y = d$, where d is a constant, is interiorized into a process (G1) for solving graphically quadratic equations of the form $ax^2 + bx + c = d$.

G2 The action of expressing equations equivalent to $ax^2 + bx + c = d$ in the form $ax^2 + bx + c - d = 0$, and then graphically solving the resulting equation using the process in G1, is an action on a process and thus promotes its encapsulation into an object conception for solving quadratic equations graphically that recognizes that the solutions of any quadratic equation in standard form correspond to the x intercepts of the corresponding quadratic function.

4. Method

Data were obtained from two different types of students: beginning university students who had studied quadratic equations as part of their algebra pre-university preparation and who had just taken a precalculus course in which they again studied these equations, and more advanced students who had further benefitted from two semesters of single-variable calculus and were then taking a multivariable calculus course at the same university. Multivariable calculus students were used since they were the most advanced group of students whose participation was easy to arrange before they went on to take specialized courses in their own science and engineering departments. Further, one of the researchers had already made repeated informal observations of student difficulties with properties of the square root while teaching the course. Using this group of more advanced students stresses the importance of some of the mental constructions conjectured in the genetic decomposition since these students should be expected to fully understand quadratic equations and all the mental constructions described in the genetic decomposition. The interview questionnaire and the written instrument used in the study were designed to give us information about which of the conjectured mental constructions students can or cannot do. Results from the study can be used to modify the teaching approach used in instruction as they will suggest the need to design and implement activities to help students make some specific mental constructions.

4.1. *Students who had just taken a precalculus course*

Eight beginning college students from different majors at the University of Puerto Rico at Mayagüez were interviewed when solving problems involving quadratic equations in order to test portions of the genetic decomposition by verifying if they could do the conjectured mental constructions and finding conjectured mental constructions in which they had difficulty. The students had just finished taking a precalculus course where they studied quadratic equations and quadratic functions prior to their participation in the study. Teaching was done by the lecture–recitation method in which class time is mainly devoted to lectures and students work on their own outside of class on assigned problems chosen from the textbook. Instruction followed very closely a traditional text [18] and syllabus. None of the researchers taught the course that particular semester. In particular, the genetic decomposition was not used to guide instruction or assign work. The students were selected based on their course grade, so that two of them were expected to be above average, four average, and two below average. Students with different course grades were chosen in order to obtain a wide range of observations. The distribution of students in these course grade groups attempts to mirror the expected normal distribution of course grades. Also, prior experience had taught us that it is from the group of average students that the most interesting observations are to be expected.

An instrument was designed to conduct semi-structured interviews with students and test their understanding of the components of the genetic decomposition. Each interview took from 45 to 70 minutes. The interviews were recorded, transcribed, analysed independently, and scored independently, by each of the researchers, and discussed as a group until consensus was reached. Data analysis considered whether students showed evidence of the processes described in the genetic decomposition. The written work was kept for reference.

The interview questions were the following:

(1) Find all the solutions of the following equations:
  (a) $(81 - x)^2 = 81$,
  $x^4 - 3x^2 + 2 = 0$.

Problem 1a had the purpose of probing students' mental constructions needed for solving a perfect square using the square root. Observe that the square root method is the most convenient way to solve the problem (it is also reasonable to use factoring as a difference of squares) since if students choose to expand they would end up with the equation $x^2 - 162x + 6480 = 0$ which is harder to solve using other methods. This problem can potentially give information about the mental constructions which are part of the genetic decomposition as described in SR1, SR2, and SR3.

In Problem 1b, students needed to realize that they can solve it applying the techniques for quadratic equations (hence obtaining information about QF3). The roots of the intermediary quadratic equation (obtained by substituting, say, $u = x^2$) could have been found by using different methods. The intention was to observe if they knew how to solve 'hidden quadratics equations' and to observe if students, especially those who did not use the square root method in the first problem, were able to use the square root method when they had to solve $x^2 = 1$ and $x^2 = 2$(as in SR2). The problem also gives information of students' use of the factoring technique (F1; the easiest method in this case) and/or the quadratic formula (QF2), and their preference for using one or the other method.

(2) Use the quadratic formula to find all the solutions of the following equation: $4x^2 + 6x + 2 = 0$.

This problem gives information about students' use of the quadratic formula, as conjectured in QF2.

(3) Solve by using different methods (at least three) to find the solutions of the following equation: $(x - 4)(x + 4) = (x - 4)$.

This problem gives us two ways to see if students can do the coordination described in F1. Students having an object conception of factoring may be conjectured that would be able to imagine subtracting $x - 4$ on both sides and then factoring without having to explicitly do it. However, failing to do that, some students would choose to expand, put the resulting equation in standard form, and then factor, applying the process described in F1. Further, the problem enables us to observe if the students divide by $x - 4$ without considering that x could be 4. The problem also gives another opportunity for observing the process in QF2. Since students were asked to use three different methods, this could give information on the mental constructions necessary for completing the square as described in SR4 and SR5. It also gives information about the preferred method by the students, the most frequently used method, and if they are flexible when solving an equation by different methods.

(4) Find all solutions of the following equations:
(a) $6x^2 - 28x - 32 = 16$,
(b) $(3x + 2)^2 + 8(3x + 2) + 12 = 0$.

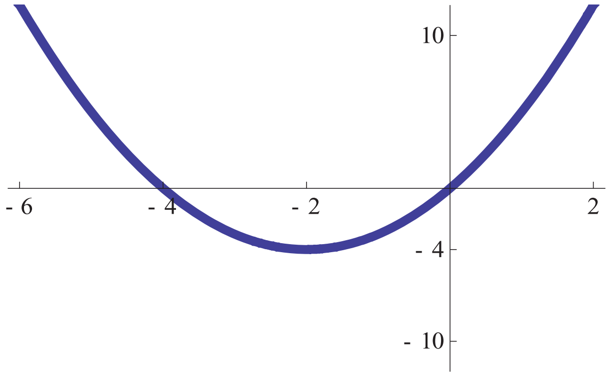
Problem 4a allows further observations of students solving quadratic equations by factoring (F2) or by the quadratic formula (QF2).

Problem 4b had the purpose of probing if the students were able to recognize a 'hidden quadratic equation' by the substitution of $3x + 2$ with another variable to obtain a quadratic equation. This provides an opportunity to observe if students may have an object conception of the quadratic formula as described in QF3. It also gives further opportunity to observe students' mental constructions when solving a quadratic equation by factoring (F1) or by the quadratic formula (QF2) as they would need to solve the resulting quadratic equation.

(5) Find all solutions of the following equation by completing the square and using the square root: $x^4 + 2x^2 - 48 = 0$.

This question had the purpose of probing if the students were able to use the method of completing the square correctly, thus potentially allowing observations into the mental constructions described in SR1, SR2, SR3, SR4, and SR5. It also gives another opportunity to observe the mental construction described in QF3 for solving 'hidden quadratics'.

(a) $x^2 + 4x = 10$,
(b) $x^2 + 4x = -4$,
(c) $x^2 + 4x = -10$.



(6) The graph of $p(x) = x^2 + 4x$ appears below. Use the graph to find how many solutions each of the following equations has. Explain.

This question allows observations on students' construction of the Process in G1.

4.2. *Science and engineering students in a multivariable calculus course*

Two questions requiring the correct use of properties described in SR1 and SR2 ($\sqrt{y^2} = |y|$ and for non-negative values of c, $w^2 = c \Rightarrow w = \pm\sqrt{c}$) were given to 121 science and engineering students as part of a test in an introductory multivariable calculus course. Students did not have access to a computer or graphing calculator during the test. These students were enrolled in three sections of a course taught by one of the researchers and can be considered to have been randomly selected.

Question 1:

Let $f(x, y) = \sqrt{y^2}$. Draw the graph f as carefully as you can. Add a verbal description if necessary.

This question inquires into students' knowledge of the property $\sqrt{y^2} = |y|$ described in SR1.

Question 2:

Let $f(x, y) = (y - x)^2$. Draw a contour diagram for f. Your contour diagram must include the corresponding z values. Use as many contours as necessary to understand the behaviour of the function. Show all your works.

This question requires students to solve equations of the form $(y - x)^2 = c$, where c is a non-negative constant. A way to solve this equation is by applying processes SR1 and SR2 to obtain $(y - x)^2 = c$ and so $\sqrt{(y - x)^2} = \sqrt{c}$. By property SR1, $|y - x| = \sqrt{c}$, so that by property SR2, $y - x = \pm\sqrt{c}$, concluding that $y = x \pm \sqrt{c}$.

5. Results

5.1. *Results from science and engineering students in multivariable calculus*

The results from this group of students are based solely on their written response to the two questions above. These questions mainly test student understanding of properties SR1

	Total number of students	Correct use of processes in SR1 and SR2	Incorrect use of processes in SR1 and SR2	No use of processes in SR1 and SR2
Question 1	121	18 (14.9%)	53 (43.8%)	50 (41.3%)
Question 2	121	20 (16.5%)	40 (33.1%)	61 (50.4%)

and SR2. Information about mental processes obtained using solely a written instrument can only be tentative. A student who when answering these questions applied correctly the properties described by processes SR1 and SR2 could conceivably have done so using only memorized information, and hence may only have had an action conception of the properties in SR1 and SR2. One may safely infer that a student who showed the incorrect use of those properties did not have the mental processes described in SR1 and SR2. It is also reasonable to infer that students not using those properties were likely not to have a process conception of the properties (they are needed to solve the problems) although they might perhaps have failed to answer the questions correctly due to other reasons (they might not know what is the graph of a function of two variables or how to obtain a contour diagram of such a function). These students either gave no answer to the test question (even though they were among the first questions asked in the test) or attempted to use another strategy (such as a point-by-point evaluation when drawing the graph of the function in Question 1). Given the above discussion, one may conclude (as suggested by the following table) that even science and engineering students who have reached a multivariable calculus course seem, for the most part, not to have the process conceptions described in SR1 and SR2. This, in turn, is reflected in students' difficulties solving equations by using the square root and completing the square. Among the incorrect use of the properties in SR1 and SR2, the most common ones were $\sqrt{y^2} = y$ and $\sqrt{y^2} = \pm y$.

The above results show the necessity of giving students the opportunity to interiorize the processes in SR1 and SR2 by designing activities which induce reflection on these specific properties. The strategy of simply stating the property that $w^2 = c \Rightarrow w = \pm\sqrt{c}$, and then just expecting students to use it correctly from then on, is not sufficient as this amounts to promoting an action conception of the formula as a memorized fact. In its place we propose to use the strategy of 'doing the same thing on both sides', as suggested by Tall, Lima, and Healy [12] and thus start with the lengthier procedure of applying the square root to both sides of the equation $w^2 = c$ to obtain $\sqrt{w^2} = \sqrt{c}$, and then from here obtain $|w| = \sqrt{c}$, from where finally $w = \pm\sqrt{c}$. Students who have interiorized this chain of actions into a Process will be able to start with $w^2 = c$, imagine taking square roots and eliminating the resulting absolute value sign (without having to explicitly do so) in order to end up with $w = \pm\sqrt{c}$. Observe that the above process would take the place of 'cancellation' techniques frequently taught by teachers (where students 'cancel' the exponent 2 with the square root) that can lead to erroneous answer (since the negative solution $w = -\sqrt{c}$ is lost). This 'cancellation' technique would be discouraged by the consistent application of the processes in SR1 and SR2 and activities designed to reflect on them. Similarly, the blind application of rules of exponents without a clear understanding of subjacent hypotheses can also lead to erroneous conclusions as would be the case in the following argument: for $c > 0$, $y^2 = c \Rightarrow (y^2)^{\frac{1}{2}} = c^{\frac{1}{2}} \Rightarrow y = \sqrt{c}$; observe the argument is valid only when $y \geq 0$. So, activities must require that students use the square root in either radical or exponential notation.

$$(81-x)^2 = 81$$

$$\sqrt{(81-x)^2} = \sqrt{81}$$

$$81 - x = 9$$

$$-x = 9 - 81$$

$$-x = -72$$

$$\boxed{x = 72}$$

Figure 2. ‘Sample student’s written response’.

5.2. *Results from students who had just taken a precalculus course*

5.2.1. *Square root and completing the square*

In Problem 1a, only three of the eight interviewed students used the square root method to solve the given equation. All of these three students found only one solution to the equation, making the same mistake. They all expressed $(f(x))^2 = a$ as $\sqrt{(f(x))^2} = \sqrt{a}$ and then concluded that $f(x) = \sqrt{a}$. An example of this can be seen in Figure 2.

These students had not interiorized the processes in SR1 and SR2 and, consequently, they had not encapsulated the coordination of these processes into an object conception to solve perfect squares by using the square root method, as specified in SR3. Note that otherwise, students could have proceeded as follows: $(81-x)^2 = 81 \Rightarrow \sqrt{(81-x)^2} = \sqrt{81} \Rightarrow |81-x| = 9 \Rightarrow 81 - x = \pm 9 \Rightarrow x = 90$ or $x = 72$. Specifically, the problem arises when the students do not recognize that$\sqrt{(f(x))^2} = |f(x)|$. None of the students interviewed solved this problem correctly. Note that from the point of view of a student who is working procedurally, that is, limited to an action conception, obtaining the right-hand sides of $\sqrt{3^2} = 3$ and $\sqrt{x^2} = |x|$ would seem like two different ways of treating expressions having the same form $\sqrt{y^2}$. Indeed, many students even think that $\sqrt{x^2} = \pm x$ is correct, and they are also bound to confuse $\sqrt{y^2}$ with $(\sqrt{y})^2$. Hence, the mere mention of the properties in SR1 and SR2 without a deeper exploration is bound to result in an action conception rendering many students unable to do required symbolic computations in other courses. An example of a student having an action conception of the property that if $x^2 = b$ then $x = \pm\sqrt{b}$ is Jennifer. She depends on using a memorized instruction (that in this case happens to be incorrect):

> Interviewer: Let me ask you, you had $x^2 = 6$, how did you get to the conclusion that $x = \pm\sqrt{6}$?
>
> Jennifer: Because when you take a square root any answer will have two solutions.
>
> Interviewer: So for example, the square root of 9, what is the square root of 9?
>
> Jennifer: plus or minus three.

We observe that two students who had difficulty applying the square root in Problem 1a showed no such difficulty when solving $x^2 = 2$ and $x^2 = 1$ in Problem 1b, stressing

5) $x^4 + 2x^2 - 48 = 0$ $\quad x^2 = w$

$w^2 + 2w - 48 = 0$

$w^2 + 2w + 1 = 48 + 1$

$w^2 + 2w + 1 = 49$

Figure 3. Karla's written work in Problem 5.

the importance of having an object conception of solving a perfect square as described in SR3.

If, as suggested by the results on Problem 1a, students do not have an object conception of solving a perfect square (as in SR3), it is not reasonable to expect them to recognize or remember the main idea behind the method of completing the square. These students can be expected to show difficulty with the processes and object described in SR4, SR5, and SR6. Indeed, in Problem 5, only two of the eight students were able to complete the square as required and the other students, like Yolanda, had no recollection of the method or even the reason for completing the square.

> Yolanda: Completing the square, it would be... that thing of completing the square I truly don't remember. I know it had something to do with the 48 that I divided it, but no, that gave me a lot of difficulty in the course.

Another student, Karla, was able to start correctly the procedure of completing the square but, not understanding the reason for the procedure, she was unable to finish (see Figure 3):

> Karla: Ok, what I understand is that the coefficient of the *w*, you have to take the square root of the coefficient and raise it to the 2, which makes no sense, really, because you are doing exactly the same, but this is what I remember you had to do. Let me see if there's another way because I remember you had to raise it to the 2... Ah! You take half of that, now I remember, very well Karla! You take half the coefficient which is $2/2 = 1$ and you raise it to the 2 and you add it to both sides. Well, you get $w^2 + 2w + 1 = 49$. Well, let me see. Here I don't know if I have to subtract, here, my doubt is if I have to subtract 49 from 1 and then solve it like that or...

> Interviewer: What is the main reason for completing the square?

> Karla: Isn't it to solve for *x* (laughs) and find a solution?

It seems that Karla had not interiorized the action described in SR4 into a process for completing the square as she did not realize the memorized procedure she used has the finality of obtaining a perfect square (so that she can then use the square root as in SR3). Not having the process in SR4 she was not able to act on that process in order to solve the equation as described in SR5. We should also mention that in Problem 3, which required students to use three different methods to solve the given quadratic equation, only one student attempted (unsuccessfully) to use the method of completing the square.

5.2.2. *Quadratic formula*

In Problem 2, students were asked to use the quadratic formula to solve the given equation. Six of the eight students remembered the quadratic formula correctly. Also, in Problem 3, after expanding and expressing the equation in the standard form, five students applied the quadratic formula to solve the equation. Further, in Problem 1b, five of the eight students identified this as a hidden quadratic. In general, most students provided evidence of having interiorized the action of applying the quadratic formula to solve equations equivalent to $ax^2 + bx + c = 0$ into a process to solve quadratic equations using the quadratic formula, as conjectured in QF2.

On the other hand, in Problem 4b, only one of the eight interviewed students recognized the given equation, $(3x + 2)^2 + 8(3x + 2) + 12 = 0$, as a hidden quadratic. Considering that in each of Problems 1b and 3, five of the eight students managed to make the corresponding substitution, $u = x^2$; this suggests that students should experience different types of functions $f(x)$ when practising the coordination suggested in QF3 of the genetic decomposition. It also suggests that the majority of students do not have an object conception of the quadratic formula that they can use in different kinds of contexts.

5.2.3. *Flexibility*

It was observed that most students did not show much difficulty with either the quadratic formula or factoring. Indeed, students are not given to using methods other than these two. In Problem 3, students were asked to solve the given equation using three different methods. Only two students tried to solve the equation by using three different methods (expand and factor; quadratic formula; divide by $x - 4$), four students tried to solve the equation by applying two different methods, and the remaining two students used only one method to solve the equation. Hence, this problem shows that students are not flexible using different methods to solve a quadratic equation since most of them used the same two methods (expand and factor; quadratic formula) and did not completed the required quantity of methods. None of the students subtracted $x - 4$ on both sides of the equation to use it as a common factor, none attempted using a graphical method, and only one attempted (unsuccessfully) to complete the square. Indeed, when solving $(81 - x)^2 = 81$ in Problem 1a, only three students attempted the convenient method (for this problem) of using square roots, while four students attempted the most inconvenient method (for this problem) of the quadratic formula. Further, when solving $6x^2 - 28x - 32 = 16$ in Problem 4a, four of the eight students showed evidence consistent with having a process for solving quadratic equations by factoring (as in F2), while the rest of the students preferred to apply the quadratic formula, again giving evidence of lack of flexibility, given that the coefficients in the equation were relatively large, making this an inconvenient method.

Perhaps, activities interrelating different solution methods will help students obtain an improved understanding of when it is better to use one method over another. This remains open for future investigations. The observed lack of flexibility suggests that many students have yet to construct a coherent schema for quadratic equations since their knowledge of these equations seems to consist mainly of actions and isolated processes. That is, the components of the genetic decomposition labelled SR (square root), QF (quadratic formula), F (factoring), and G (graphical) for the most part seem to be isolated in students' minds. Hence, many students would seem to be either at the intra-stage or inter-stage of schema development.

5.2.4. *Nature of the solutions*

When solving $(x-4)(x+4) = (x-4)$, three students divided both sides of the equation by $x-4$ without considering that x could be 4. None of the students could explain why he/she found only one solution. Giovanni was not be aware that dividing both sides of a given equation by an algebraic expression that could take the value of zero may result in an equation that is not equivalent (similar observations have been made by [10] and [11]):

Giovanni: We can divide the $x-4$ by the $x-4$ it turns into a 1. We have $x+4=1, x=1-4$, $x=3$ (sic).

Interviewer: Are those all the solutions?

Giovanni: So far, yes.

Our genetic decomposition assumes that students taking a precalculus course in their first year of university studies have already done the mental construction of symbolic manipulation processes enabling them to use the basic operations of addition, subtraction, multiplication, and division to 'do the same thing' on both sides of an equation and recognize the equivalence (or non-equivalence) of the resulting equation. As seen above, this is not always the case. Further, when he says 'so far' it seems as if Giovanni thought that different methods may produce different solutions. This was also observed in another student, María. It seems that these students have not explored the relationship between different solution methods for quadratic equations, that is, they seem not to have constructed processes relating the different components (SR, QF, F, G) of the genetic decomposition:

Interviewer: So you found one solution for exercise a? (Referring to $(81-x)^2 = 81$)

María: Exactly, it said to find all solutions.

Interviewer: So you think you have all solutions.

María: But if I use special products I get other solutions, right?

Perhaps, another reason for the apparent lack of awareness of the nature of the possible solutions to a quadratic equation is that many students have not explored symbolically and numerically the solutions given by the quadratic formula as suggested in part QF1 of the genetic decomposition. Further, given that the graphical context clarifies the possible nature of the solutions, many seem not to have constructed a mental process for the graphical solution of quadratic equations as described in portions G1 and G2 of the genetic decomposition. This is corroborated by the fact that only one of the eight students was able to do Problem 6 without any help. An example of a student not having interiorized the process described in G1 is María, who seemed to need explicitly doing an algebraic calculation. In Problem 6:

María: If $p(x) = 10$, so if I have to find x but using the graph... I don't remember, I don't know what I have to do... I can't do any calculations! (laughs)... I would have to factor to know.

Esmeralda is another example. In her case she seemed to need to do explicit numeric calculations thus showing she did not have the process conception in G1:

Esmeralda: Ok... I think, let me see, the first one has one solution.

Interviewer: Explain why.

Esmeralda: So, using the original equation I have to find a value of x that when I substitute will give me 10. So what I did was, for example, I took number 2, when x is 2, because if I substitute it will be $2^2 + 2 \cdot 4 = 10$, it would be $4 + 8$.

Interviewer: How did you think of that 2? Looking at the equation?

Esmeralda: Yes, I thought of the 2 because I had to look for a number that when squared and when I added something that was multiplied by 4, it couldn't be too big so that I would get 10. So I chose 2.

Jennifer seemed to think that the number of solutions of each one of the equations $x^2 + 4x = 10$, $x^2 + 4x = -4$, and $x^2 + 4x = -10$, is 2 since the graph of $p(x) = x^2 + 4x$ touches the x axis twice:

Jennifer: Ok, if we look at it the first one is supposed to have two solutions.

Interviewer: And how can you explain that to us using this graph?

Jennifer: Well, because it has two points touching the x axis.

Interviewer: Where?

Jennifer: at -4 and 0.

Interviewer: And that would be for equation a? (Referring to $x^2 + 4x = 10$)

Jennifer: Uhum, and for equation b and equation c.

Interviewer: So in the three equations we have two solutions because that graph touches the x axis twice, is that right?

Jennifer: Yes.

Jennifer had not interiorized the process described in G1 of the genetic decomposition. She seemed to have only partially memorized the action described in G2. Further, for the given function her belief that the equations will have two solutions was in accordance with her belief that a polynomial of degree n will have n roots where she seemed to expect n distinct real roots:

Jennifer: In the before mentioned exercise (Referring to the possible solutions of $x^4 - 3x^2 + 2 = 0$ before solving the equation) we have a power of 4 which means that we will have 4 solutions...

Later, in Problem 5, she again gave evidence of expecting four solutions when solving $x^4 + 2x^2 - 48 = 0$. After substituting $u = x^2$, being unable to complete the square she used the quadratic formula:

> Jennifer: These are the solutions in the case of u, or is it of x? … exactly, ok, ok, now I understood… ok, now we have the four solutions! (She made an algebraic mistake that lead her to find 4 real 'solutions'.)

Hence, Jennifer seems not to have explored the possible solutions of a quadratic equation as described in QF1 as thus seems unaware of the possibility of repeated roots and of complex roots.

Students showed lack of confidence when using complex numbers. Students showing this difficulty seemed not to have interiorized the process of recognition that the quadratic formula produces all solutions of a quadratic equation and that there can be 0, 1, or 2 solutions, as described in QF1. Such a process when coordinated with a process for complex numbers would result in a recognition and symbolic manipulation of the possible complex roots of a quadratic equation. Esmeralda is one such student:

> Esmeralda: Let me see… ok… I factor and I am left with $(x^2 + 8)(x^2 - 6) = 0$, then I'm going to set $x^2 + 8 = 0$, $x^2 - 6 = 0$. Now I'm going to solve for x… and on one side I was left with $x^2 = -8$, but I think that when I apply the square root it shouldn't be negative. I am not sure if I should take the negative from the 8 out, but if I can't then it wouldn't be a solution (She wrote $\sqrt{x^2} = -\sqrt{8}$).
>
> Interviewer: And if I tell you that solutions need not be real numbers, that other solution, could you write it some other way?
>
> Esmeralda: If I were using complex numbers… I don't know.

Yolanda was also uncertain of how to deal with complex numbers. In Problem 5, not being able to complete the square she tried to factor, and mistakenly ended up having to solve $x^2 + 2 = 0$:

> Yolanda: … It would be $x^2 + 2 = 0$ and x would be $\sqrt{2}$.
>
> Interviewer: Square root of what?
>
> Yolanda: of 2, but it would be negative, and something like this I know is not possible, I had understood that you couldn't have square roots of negatives, that's where the imaginary came in, I don't know, something like that.
>
> Interviewer: (Noting that Yolanda had only given two solutions) Why have we found all solutions? Or are some missing?
>
> Yolanda: I understand that yes…
>
> Interviewer: Let me ask you, is there supposed to be a relation between the exponent in the equation…
>
> Yolanda: Yes, that there's supposed to be 4 solutions.

Interviewer: And what do you think happened that there's only two? What happened? Where did we leave them?

Yolanda: I don't know where I put them, but there's supposed to be 4... the same with this one (Referring to $(x-4)(x+4)=(x-4)$), there's supposed to be 3, because there are 3 variables, that is, there are 3 x's... in (problem) three you could write it, like everything to the power 3, something like that, but I don't remember.

In her last statement, observe that Yolanda might be trying to suggest that the equation $(x-4)(x+4)=(x-4)$ is equivalent to an equation having a term with x^3. She seems not to be able to imagine the symbolic manipulation necessary to write the given equation in standard form (without having to explicitly do it), that is, she seems not to have a process conception for that symbolic manipulation. Hence some student difficulties seem to stem from their not having process conceptions of complex numbers and/or of processes for the symbolic manipulation of equations.

Karla was the only student able to do Problem 6 without any help:

Karla: The 10, I understand you have to find the position of x using the 10, in this case as the y position, and that gives you the possible solutions of x which are 2. In the second, 1 possible solution and in the -10 none.

Karla seems to have the mental processes described in G1 and G2 as she is able to solve the problem in her mind without making any explicit computation.

The above results again suggests that many students have not constructed a coherent schema for solving quadratic equations, as they show difficulty relating different solution methods, including the graphical method, when discussing the nature of the solutions. Students also show difficulty coordinating processes for solving quadratic equations with processes for complex numbers and processes for the symbolic manipulation of equations. This suggest that the genetic decomposition of the schema for solving quadratic equations needs to include processes that interrelate the different components of the schema, particularly the quadratic formula and factoring methods with the graphical method. This can potentially help students achieve at least an inter-stage of development for the schema of quadratic equations. In particular, this can potentially help students understand that solutions do not depend on the method used, can help them develop flexibility solving quadratic equations as they will have the opportunity to reflect on when using a particular solution method, can be expected to be more convenient than using another, and can also help students anticipate the number and nature of the solutions they might expect.

After each of the researchers independently analysed the students' performance on the entire interview relative to the conjectured mental constructions in the genetic decomposition and their interrelation, the researchers then discussed the results as a group. Students were initially grouped according to the scores they received on the interview questionnaire and after a more in-depth discussion of how well these students could interrelate the different components, processes, and objects of the genetic decomposition, the consensus was reached that none of the eight interviewed students seemed to have developed a trans-quadratic equations level of schema development, and that only two of them seemed to have developed an inter-quadratic equations level of schema development. All the other interviewed students could only show evidence of having a collection of mostly isolated processes and memorized procedures for solving quadratic equations and hence they were classified at the intra-quadratic equations level of schema development. We did not explore specific actions and processes that may help students to relate the different components of the proposed genetic decomposition in the present paper, but this is certainly needed for future investigations.

6. Summary and discussion

It may be argued that a conceptually simple way to solve an equation is, whenever possible, to reverse the process used to form the equation in the first place. In the case of quadratic equations, this would mean that the main idea for solving quadratic equations would be to reverse the process of squaring by somehow using the square root. In this study, we have seen that students do not regard solving a quadratic equation as a process of reversion. Even when given a perfect square, such as $(81 - x)^2 = 81$, students' tendency is not to use the square root. Part of the reason for this is that students do not understand the basic properties of the square root, as underscored by the study results obtained from the science and engineering students in a multivariable calculus course. This suggests that more time and attention should be given to the design of activities and exploration of the basic properties given by (1) $\sqrt{x^2} = |x|$ and (2) if $x^2 = a$ then $x = \pm\sqrt{a}$. We propose that it is necessary that students be given the opportunity to interiorize the 'do the same thing on both sides' step–by-step process that relates these properties: if $a \geq 0$ and $x^2 = a$ then $\sqrt{x^2} = \sqrt{a}$, so that $|x| = \sqrt{a}$, and hence $x = \pm\sqrt{a}$. Once students have a mental Process relating all these steps, they will not need to do them explicitly and will be ready to start with $x^2 = a$, supply the mental steps necessary, and end with $x = \pm\sqrt{a}$. The method of completing the square is essentially the algebraic manipulation of a quadratic equation in order to render it in a form amenable to its solution by using the square root. So, students having difficulty with the basic properties of square roots will not see the rationale for the method. This is supported by the fact that, in the student interviews, only two students were able to use this method.

For the most part, students may be able to apply the quadratic formula to solve equations; however, for many students, this is done as an action, i.e. as a memorized procedure remaining unrelated to the basic idea of using the square root or of its graphical interpretation. The fact that most students did not substitute $3x + 2$ for a new variable in Problem 4b suggests that students have not encapsulated the process of solving a quadratic equation using the quadratic formula into an object, which would enable them to use it as a tool in non-standard situations. So, in the symbolic treatment of solving quadratic equations using the quadratic formula, students can benefit from seeing a wider variety of 'hidden quadratics'. While this study only included one question (Question 6) dealing with the graphical interpretation of quadratic equations, the results obtained strongly suggest that students need more opportunity to explicitly consider the connections between the symbolic and graphical representations of quadratic equations.

One of the contributions of the present study is the proposal of a detailed conjecture of mental constructions students may do in order to understand quadratic equations and which can serve as a base for further studies of students' construction of a schema interrelating the different methods for solving quadratic equations. The genetic decomposition and results from the study also highlight two specific mental constructions (SR1 and SR2) that play a key role in students' understanding of quadratic equations, but that students have difficulty doing and seem to be overlooked in traditional instruction. Further, another result of the study underscores the importance of numerical and graphical explorations into the nature of the possible solutions of a quadratic equation.

This study is the result of a first cycle of APOS research. We intend to continue with a second research cycle in which teaching materials and more effective pedagogies to help students do the conjectured mental constructions in the genetic decomposition are developed and used in the classroom before undertaking a second set of student interviews to see if students show the expected improved performance.

Disclosure statement

No potential conflict of interest was reported by the authors.

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