Student understanding of the relation between tangent plane and the total differential of two-variable functions

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Action-Process-Object-Schema (APOS) theory and tools resulting from dialogue with the Anthropological Theory of the Didactic (ATD) were used to analyse data from semi-structured interviews and teaching materials to study students' understanding of the relationship between tangent planes and the total differential. Results of the study show students' difficulties relating these ideas and suggest a refinement of the initial genetic decomposition. They also underline aspects of the teaching materials that need to be considered to promote those constructions and development of a complete praxeology for the total differential. This study exemplifies how the dialogue between a cognitive theory and one that focuses on institutional aspects of mathematics education, can provide tools to deeply analyse the teaching and learning of a mathematical topic.

Keywords: APOS, Anthropological Theory of the Didactic, calculus, differential, moments of study, tangent plane

La théorie Action-Processus-Objet-Schéma (APOS), ainsi que les résultats d'un dialogue avec la Théorie Anthropologique du Didactique (ATD), ont été appliqués à l'analyse d'entretiens semi-structurés et de manuels, afin d'étudier la compréhension, par les étudiants, de la relation entre plans tangents et différentielle totale. Les résultats de l'étude montrent que les étudiants éprouvent des difficultés à relier ces idées et suggèrent des manières d'affiner la décomposition génétique initiale. Ils permettent également d'identifier des aspects des supports d'enseignement qu'il s'agit de prendre en compte pour faciliter la construction de ces notions et le développement d'une praxéologie complète pour la différentielle totale. Cette étude illustre la fertilité d'un dialogue entre une théorie cognitive et une théorie centrée sur les aspects institutionnels en didactique des mathématiques, pour s'outiller en vue d'analyser l'enseignement-apprentissage d'un sujet donné.

Mots-clé: APOS, Théorie Anthropologique du Didactique, analyse, différentielle, moments d'étude, plan tangent.

INTRODUCTION

Functions of several variables play a very important role in mathematics and the applied sciences. In the last ten years, a number of publications have discussed the geometric and formal understanding of two-variable functions. These include Martínez-Planell and Trigueros (2012), Trigueros and Martínez-Planell (2010), and Weber and Thompson (2014). However, there is little published work dealing with the understanding of the differential multivariable calculus, and more specifically, of the total differential of two-variable functions.

As observed by Allendoerfer (1952), how best to introduce the notion of differential can be controversial. Indeed, Spivak (1965, p. 45), who used the notion of differential forms to present a modern treatment of the differential in his classic textbook "Calculus on Manifolds", recognized that "It is a touchy question whether or not these modern definitions represent a real improvement over classical formalism." Since we are concerned with the cognitive analysis of students' constructions of the total differential we decided that an approach closer to the first approach in the epistemological development of the concept, as used in introductory calculus books (e.g. Stewart, 2006), was pertinent to avoid the controversy. We thus restrict our attention to the total differential and use the classical notation $df = (\partial f / \partial x)dx + (\partial f / \partial y)dy$, where *dx* and *dy* are defined as independent variables. Tall (1992), applied this approach using local linearity as a powerful idea to give coherence to many of the ideas of calculus. He introduced a geometric model (similar to Figure 1) to write the total differential of a two-variable function $dz = (\partial z / \partial x)dx + (\partial z / \partial y)dy$ as $dz = (\partial z_x / \partial x)dx + (\partial z_y / \partial y)dy$, where dz_x is the vertical change on the tangent plane corresponding to a horizontal change dx (leaving y fixed) and similarly for dz_y. He argued that the partials are quotients of lengths which may be cancelled and that this is more meaningful than introducing the symbol ∂z in the terms $(\partial z/\partial x)$ and $(\partial z/\partial y)$. A similar approach was used by Martínez-Planell, Trigueros, and McGee (2015), who applied APOS to study different components of the differential calculus of these functions: partial derivatives, planes, tangent planes, directional derivatives, and total differential. The present report expands their discussion of student understanding of the total differential and its relation to the idea of tangent plane. In this study some results from the dialogue between APOS Theory (Action-Process-Object-Schema) (Arnon et al, 2014) and Anthropological Theory of Didactic (ATD) (Chevallard, 1999), where both theories were critically discussed in terms of their components, play an important role.

In what follows we first describe the theoretical background of the study and a preliminary genetic decomposition, then we discuss the methodology used to obtain and analyse the data. We then report on the analysis of the teaching materials and students' mental constructions. We finish by discussing the results obtained, suggesting changes to the preliminary genetic decomposition, and a brief conclusion.

THEORETICAL BACKGROUND

Our theoretical background consists mainly of APOS, together with the ATD, particularly some results emerging from the dialogue between APOS and ATD described in Bosch, Gascón, and Trigueros (2016). These theories take a different stance in respect to Mathematics Education. APOS is a cognitive theory that studies how students learn by proposing detailed models of their possible constructions while ATD focuses on the role that institutions, in a wide sense, play in the teaching and learning of mathematics. An open but critical dialogue was undertaken taking into account the different aspects of both theories to find out if each could provide a new point of view on the other without violating their main hypothesis.

In APOS (Arnon, et al, 2014), an Action is a transformation of a mathematical object that is perceived by the individual as external. It may be a rigid application of an explicitly available algorithm or of a memorized fact or procedure. As an individual repeats and reflects on an Action, it may be interiorized into a Process. A Process allows the individual to reflect on the steps of the Process, omit steps, and anticipate the result without having to explicitly perform the Process. A Process may be coordinated with other Processes. As an individual needs to apply actions on a Process, he/she may come to see the Process as an entity in itself. When the individual can perform, or imagine performing Actions on a Process it is considered that the Process has been encapsulated into an Object. An Object may be de-encapsulated into the Process it came from as needed in problem situations. A Schema for a specific mathematical idea is a coherent collection of Actions, Processes, Objects, and other previously constructed Schemas that are related to the mathematical idea. Schemas will not be further discussed since we have not used dialogue results referring to them.

In APOS, research on student understanding of a specific mathematical concept starts by establishing a conjecture, called a genetic decomposition (GD), describing mental constructions that students may do in order to construct the concept. The GD may be based on different factors including the mathematics itself, classroom experience, and literature results. A GD is not unique and needs to be validated by experimental data from students. What typically happens is that a preliminary GD is proposed and the data obtained (usually through semi-structured interviews) show that students make unconjectured mental constructions and show difficulties with some of the conjectured constructions. This information leads to refining the GD so that it reflects the constructions that students actually do, and to develop activities to help students make the conjectured mental constructions. The classroom implementation of a GD may be done using the ACE cycle (Activities, Class discussion, Exercises), an instructional approach that supports the mental constructions called for in the GD.

In ATD the activity of mathematics and its study are considered parts of human activity in social institutions (Chevallard, 1999). The theory considers that any human activity can be explained in terms of a system of *praxeologies*, or sets of practices which in the case of mathematical activity constitute the structure of what are called *mathematical organizations* (MO). MO always arise as response to a question or set of questions. In a specific institution, one or several techniques are introduced to solve a task or a set of tasks. Tasks together with the associated techniques form what is called the *practical block* of a praxeology. The existence of a technique inside an institution is justified by a technology, where the term "technology" is used in the sense of a discourse or explanation of a technique. The technology is justified by a *theory*. A theory can also be a source of new tasks and techniques. Both, technology and theory constitute the *technological-theoretical block* of a praxeology. Thus, a praxeology is a four-tuple (tasks, techniques, technologies, theories), consisting of a practical block, and a theoretical block. Typically, a praxeology gives rise to other praxeologies as different problems are explored, techniques are generalized and new ones are introduced, the range of application of technologies expand, and the theoretical basis grows to encompass more general phenomena. This results in MOs consisting of interrelated sets of praxeologies.

As part of the movement to compare different theories and seek complementarities or points of contact (Bikner-Ahsbahs and Prediger, 2014), a dialogue between APOS theory and the ATD was conducted. Results of the dialogue allowed to reduce the distance between these theories by proposing new research tools that enrich both so that one theory can complement the other without violating its main tenets (Bosch, Gascón, and Trigueros, 2016). When considering their respective theoretical components, a parallelism was noticed between the GD notion of APOS and that of Reference Epistemological Model (REM) in ATD. The notion of "generic student" of the educational institution was developed to incorporate the institutional dimension in APOS, and the personal dimension in a given institution and its praxeological equipment in ATD. The dialogue starting from the technical and technological components showed that a reinterpretation of each theory's results and methodology was possible and that it could benefit both theories. In this study, we use two of the proposed tools emerging from that part of the dialogue: It was considered that APOS' conception types (action, process, and object) can be reformulated to describe the dynamics of the articulation of praxeologies as the technical work is developed through successive aggregations and organized around a common theory. Thus, the notions of action-technique (memorized, rigid, and/or applied to isolated tasks), process-technique (supported by a technological-theoretical discourse, presenting variations, connected to other techniques) and object-technique (taken as an object of study) in an institution were defined; it was also concluded that the description of the teaching and learning processes can benefit from praxeological analysis and the notion of the didactic moments of study of the ACE cycle was developed in order to use them to describe and organize the teaching and learning process based on the constructions included in a GD in an institution. The moments of study of the ACE cycle: Moment of the first encounter, where tasks are proposed in terms of Actions on already constructed objects; Moment of exploration of types of tasks, where Actions may be interiorized into Processes or Processes can be coordinated in the emergence of techniques; Technological-theoretical moment, where explanations, justification, and Schemas and new theory are developed; Moment of practice with the techniques, where Processes are constructed, coordinated, reversed, and encapsulated into Objects, and new Actions on those Objects can be proposed to study, for example, its properties; Moment of institutionalization, where material that will be used in subsequent study is distinguished from that which only served a pedagogical purpose; and Moment of evaluation. It is important to clarify that the order of the moments is not fixed. It depends on the didactical organization of a given institution but, independently of their order, it is expected that the teaching activity creates adequate didactical situations where all the moments of study appear, thus attaining didactical "balance" and promoting an effective study process.

PRELIMINARY GENETIC DECOMPOSITION

We show the preliminary genetic decomposition (GD) proposed for the concepts of plane, tangent plane, and total differential. This GD guided the development of the interview instrument for this study. The portion of the GD shown in this article only considers the mental construction of tangent planes to graphs of functions $z=f(x,y)$, i.e. we restrict our attention to surfaces of the form $z=f(x,y)$.

Plane

Given a non-vertical plane, Processes of slope of a line and fundamental plane (planes of the form $x=c$, $y=c$, $z=c$, for c constant) are coordinated into new Processes of vertical change in the x and y directions, where it is recognized that vertical change in the x direction can be described as a function of the horizontal change in the x direction ($\Delta z_x=m_x\Delta x$), and similarly for vertical change in the y direction ($\Delta z_y=m_y\Delta y$). These Processes are coordinated into a Process of total vertical change on a plane in three-dimensional space so that total vertical change in any plane is given in terms of the sum of vertical changes in the directions of the coordinate axes: ($\Delta z=\Delta z_x+\Delta z_y=m_x\Delta x+m_y\Delta y$; see Figure 1). The need to perform Actions which are treatments and conversions in and between representations (Duval, 2006) on the Process of total vertical change promotes its encapsulation into the plane in three dimensions as an object. In particular, the equation $z-z_0=m_x(x-x_0)+m_y(y-y_0)$ can be seen as the vertical change on a plane with slopes m_x and m_y from an initial point (x_0,y_0,z_0) to a final generic point (x,y,z) and is also the equation of a plane that contains the point (x_0,y_0,z_0) and has slopes m_x and m_y (point-slopes formula for a plane).

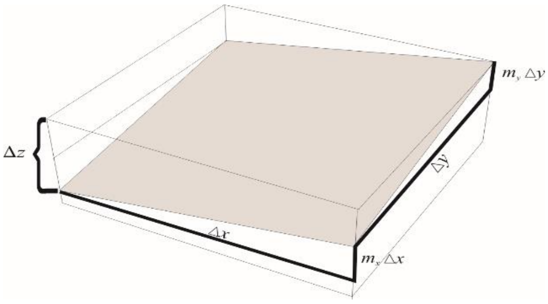

Figure 1: $\Delta z=\Delta z_x+\Delta z_y=m_x\Delta x+m_y\Delta y$

Tangent Plane

The Process of partial derivative is coordinated with that of non-vertical plane into a new Process where tangent planes to any surface of the form $y=f(x,y)$ at different points can be considered and computed. When there is a need to perform Actions on particular tangent planes to describe the surface in terms of behaviour associated with its tangent plane(s), this Process is encapsulated into the tangent plane Object.

The Total Differential

Treatment and conversion (Duval, 2006) are performed as Actions (Trigueros and Martinez-Planell, 2010) on the tangent plane Object to recognize it as the differential. This construction of the differential involves the construction of dx and dy as Objects. These Objects play the role of independent variables rather than infinitesimal objects.

To summarize, the above GD essentially proposes that students first do the mental construction of the Process of total vertical change on a plane: $\Delta z=\Delta z_x+\Delta z_y=m_x\Delta x+m_y\Delta y$. Then they coordinate this Process with a Process of partial derivative to obtain a Process of tangent plane at $(x_0,y_0,f(x_0,y_0))$: $z-f(x_0,y_0)=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$. Students then do Actions of notational change and of geometric interpretation on the Process of tangent plane and interiorize these Actions into a Process of total differential: $[df(a,b)](dx,dy)=f_x(a,b)dx+f_y(a,b)dy$.

METHOD

An instrument consisting of six questions was prepared to test student understanding of the different components of the GD. We only report on three of these questions. The instrument was used in semi-structured interviews with 26 students who had just finished a multivariable calculus course. These students were chosen from three sections that had different professors. Section T (9 students) used a traditional textbook (Stewart, 2006; this text interprets dx and dy as real valued variables) and syllabus with all the homework problems chosen from the text. Section E1 was an experimental section (9 students) using the same textbook but complemented with a set of activities designed to help students make the mental constructions described in the GD. Section E2 was another experimental section (8 students) using the same textbook with extra activities for planes and tangent planes but not for the total differential. In this section the total differential was defined but was not discussed in class. All sections had the same set of textbook homework problems for the total differential. All three professors were experienced (over 20 years teaching), having taught the course many times, and popular with students (as judged by student evaluations). Each of the professors was asked to choose 3 above average, 3 average, and 3 below average students to provide a balanced distribution. One student did not show up. The interviews lasted from 40 to 60 minutes. As they were interviewed, students also produced written answers. The interviews were recorded, transcribed, individually analysed, and then discussed as a group and results were negotiated among the researchers. The questions of interest are:

Problem 1. Students were given the plane below and asked to find the slopes in the x and y directions (m_x,m_y), the total vertical change (Δz) for Δx=4 and Δy=5, and the equation of the plane. (Note m_x=3, m_y=1, and if Δx=4 and Δy=5 then $\Delta z = m_x \Delta x + m_y \Delta y = 3(4)+1(5)=17$. Also, the equation of the plane is $z-2=3(x-1)+1(y-2)$.)

Problem 2. Students were given the graph below and were asked for the sign (positive, negative, zero) of $\partial f/\partial y(4.0,0.7)$ and $D_{<-2,1>}f(4,0)$. (Note that $\partial f/\partial y(4.0,0.7)<0$ and $D_{<-2,1>}f(4,0)>0$.)

Problem 3. The following plane is tangent to the graph of $z=f(x,y)$ at the point (1,2,0). Find the differential of f at the point (1,2), $df(1,2)$. (Observe that since m_x=1 and m_y=3 then $df(1,2)=1dx+3dy$.)

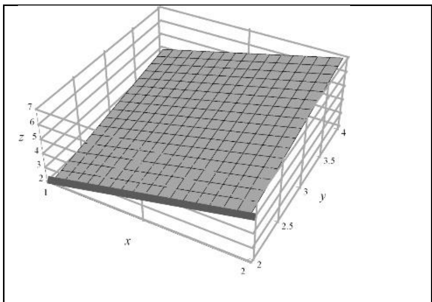

Figure for problem 1

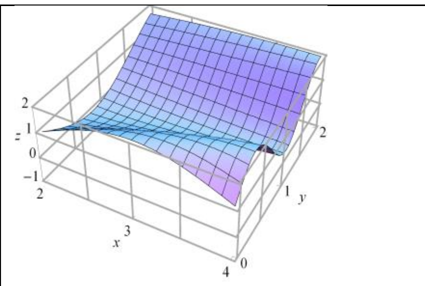

Figure for problem 2

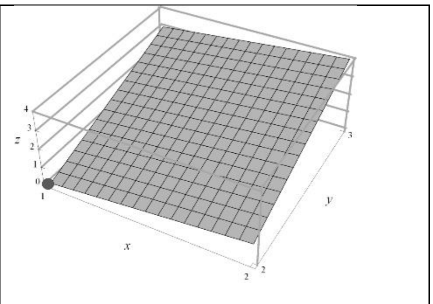

Figure for problem 3

ANALYSIS OF TEACHING MATERIALS IN TERMS OF THE RESULTS OF THE APOS-ATD DIALOGUE

Considering students' learning from a broad point of view, it is important to recognize that institutional and cognitive aspects play complementary roles. Therefore, using the tools emerging from the APOS-ATD dialogue can give a comprehensive picture of practices needed in the institution. We used two of the tools that emerged from the APOS-ATD dialogue, namely, the notion of action-, process-, and object-techniques, and the moments of study of the ACE cycle, to analyse how the textbook used (Stewart, 2006) and the activity sets contribute to the development of the generic student personal praxeologies.

Analysis using the moments of study of the ACE cycle

The *moment of the first encounter* occurs twice in the textbook. In section 12.5 planes are introduced by tasks regarding the point-normal equation as a technique with no mention of the notion of vertical change on a plane nor the point-slopes equation of a plane or any related tasks, so the corresponding techniques are not developed. When the tangent plane is introduced in section 14.4, the point-slopes equation is derived but no tasks are introduced to make apparent its geometric meaning. In that same section, the differentials *dx* and *dy* are defined as independent variables, in the sense described in the introduction, and are used to define the total differential. The geometric interpretation of the total differential is limited to a drawing intended to motivate the idea that *dz* may be used to approximate Δz. There is no explicit discussion or representation of $\Delta z_x = m_x \Delta x$ and $\Delta z_y = m_y \Delta y$ in the 3D context, so the formula for the total differential remains geometrically unmotivated. The ACE-cycle activity sets allow for an early introduction to the point-slopes equation of a plane that emphasizes its geometric meaning. However, the first encounter with the total differential stresses symbolic computation rather than geometric understanding of its relation with the tangent plane.

Corresponding to the *moment of task exploration*, no tasks exploring the relation between the concepts of vertical change and the point-slopes equation of a plane, including the tangent plane, were found in the textbook. The tasks to explore the relation between the tangent plane and the total differential are computational, without any attempt to interpret the geometry of either the point-slopes equation of the tangent plane or the differential as a vertical change. Tasks include: verifying that a function is differentiable; the symbolic computation of the linearization and total differential at a given point and their use in approximations for small changes in the domain variables, including an application to approximate change in the volume of a right circular cone. The techniques used to solve these tasks lack geometric interpretation. Thus, the tangent plane and the differential emerge as useful formulas that are somehow isolated one from the other. The ACE-cycle activity sets do not allow students' construction of the coordination of the Process of interpreting geometrically these tasks and the Process of interpreting them analytically into those Processes needed in a solid relation of tangent plane and total differential. Rather, they concentrate in relating the geometric representation of the tangent plane with the total differential, but mainly through tasks that can be done as Actions, and do not further explore the relation between other representations.

In the textbook, there are no exercises explicitly exploring vertical change on a plane, the point-slopes equation of a plane, or the geometric interpretation of tangent plane and its relation to the total differential that could be related to the *moment of practice with the techniques*. There are many problems, as in the task exploration, requiring numerical and symbolic computations, including applications to the physical sciences. Supplementing the textbook, activity sets include problems where students may construct a Process conception for the notion of vertical change on a plane and Processes to use it to obtain the point-slopes equation of a plane. However, activities do not include problems to construct an Action conception of the relation between the tangent plane and the total differential.

Related to the *technological-theoretical moment*, the idea of vertical change on a plane is not used in the textbook as a technological resource (explanation) to construct a basis for the ideas of plane, tangent plane, and the total differential. Hence the corresponding formulas emerge as symbolic expressions devoid of clear significance. Further, except for being derived from each other, they can be expected to remain relatively isolated since there are no opportunities to interiorize the notion of vertical change on a plane into a Process that can serve as a simple unifying idea to interrelate them. On the other hand, the activity sets use the notion of vertical change on a plane as an organizing idea relating the notions of plane, tangent plane, and total differential, making this an important part of the *technological-theoretical moment* and allowing for a more coherent presentation of these ideas.

In the textbook, the idea of a plane is developed into the point-normal equation, but then this equation is not explicitly related to the point-slopes equation of a plane, making it difficult to relate the ideas of tangent plane and total differential. The notions of total differential and tangent plane emerge as techniques of the differential calculus presented as formulas with some explanation in the *moment of institutionalization*. This may induce students to memorize them and to relate their application to “small values” or “sufficiently close” values. These ideas remain as such, unrelated to other ideas outside of the numeric and symbolic realm. The activity sets help the institutionalization of the use of the point-slopes equation of a plane to organize ideas of the differential calculus of two-variable functions while stressing geometric understanding.

In the review section in chapter 14 of the textbook, the concept check presented can be considered as the *moment of evaluation*. It amounts essentially to the recollection of definitions and formulas, and their use in tasks that mainly require symbolic computations. This emphasis results in students’ tendency to memorize, which is enough to respond to review questions. There are no questions inquiring about geometric understanding or the interrelation of ideas. The activity sets include evaluation questions and tasks.

Overall, using the moments of study of the ACE-cycle shows that the practical and theoretical components of the praxeologies for the differential cannot be developed, as the tasks and techniques introduced in the textbook are insufficient. In terms of APOS theory it can be expected that students can only construct an Action conception of this concepts. Memorized formulas are easily forgotten so that by the end of a term many students can be expected to have retained little or no recollection of the main ideas of the differential calculus of two-variable functions. The GD was a first step towards helping students construct a more coherent understanding of these ideas. However, the lack of detail in portions of the GD resulted in activity sets that did not render a solid complement from the point of view of the study moments.

Analysis in terms of the different types of techniques

An analysis of how the textbook (Stewart, 2006) and the GD-based activities relate the concepts of tangent plane and total differential in terms of the different types of techniques in the institution (Bosch, et al., 2016), complements the analysis using the moments of study of the ACE cycle.

The analysis of the book shows that the notion of plane is introduced as an action-technique, since it is presented as a rigid technique with no variation and related to specific tasks that do not allow linking it with other techniques. The notion of tangent plane to a surface is derived in terms of the point-slope equation of the plane, but there are no tasks that may contribute to its definition or relate it to the equation of the plane presented before. The equation for the tangent plane remains isolated so it can also be considered an action-technique. The same happens when the total differential is introduced by means of a formula and tasks consisting of Actions to calculate it in different situations, and its relation to the tangent plane is introduced by a variety of computational isolated tasks; it is presented as an action-technique. Through the problem sections and chapter summary, tasks emphasize the use of symbolic expressions to calculate tangent plane and differential. They are not linked to each other and may be used by students as a sequence of memorized Actions. Since the observed teachers in the institution follow the textbook closely it may be concluded from this analysis that students will be conditioned to use them as action-techniques.

The activities designed with the genetic decomposition have the goal of helping students make sense of the tangent plane and total differential as interdependent concepts that can be used in the solution of problems. An analysis in terms of the types of techniques developed in the activities contributes to assess their potential to aid in the articulation of praxeologies that can help students make sense of these notions and examines whether some process-techniques need to be developed into object-techniques by making them objects of study in themselves.

The activity sets start with an early introduction to the point-slope equation of a plane as a process-technique which includes the relation between its analytical and its geometric meaning and involve tasks that promote reflection and interpretation of the technique. They also include tasks to reflect on the notion of vertical change and its use in deducing the point-slopes equation of the plane that will later on support the relation to the total differential that can also be considered as process-techniques. The total differential is introduced by an action-technique which stresses symbolic computation. Techniques relating tangent plane and total differential consist mainly of tasks where students link vertical change with the total differential. These remain as action-techniques since they are limited to the graphical representation of tangent plane and the relation with the point-slopes equation of a plane is not explored. Overall, the activity sets are designed taking the notion of vertical change on a plane as an organizing idea to relate the notion of plane, tangent plane and total differential. They introduce process-techniques that include geometrical representations and connections with previously introduced techniques that permit interpretation and justification by a theoretical-technological discourse. However, the activity sets do not contain activities for students to reflect on the technique of total differential as an object-technique, its relation to other techniques or its validity. For example, the fact that $df(a,b)$ is a function of two independent variables (dx,dy), the dependence on (a,b), its graph as a translation of the tangent plane, and its relation to the definition of differentiability are not explored.

This analysis evidences that in the textbook used in the course, the moments of study are unbalanced (i.e. some moments and or ideas are not well represented) and may not foster a deep understanding of the total differential and its relation to the tangent plane. The analysis of the different types of techniques indicates that techniques introduced in the textbook are constrained to action-techniques which may limit the understanding of the concepts introduced and their relations. Both analyses show that the activity sets have some potential to foster the construction of the concepts at stake. However, it was found that the moments of study are not well balanced so the activity sets need to be redesigned. The analysis through the types of techniques indicates that it is necessary to develop the total differential into an object-technique to enhance the potential of the activities to promote the intended praxeologies.

ANALYSIS OF STUDENTS' CONSTRUCTIONS

None of the interviewed students clearly exhibited a process conception of the total differential. Only one student showed an emergent Process conception, 6 showed an Action conception, and the other 19 students showed no knowledge or recollection of the concept.

Most students seemed to depend entirely on a symbolic representation of the differential, to the extent that seeing the "*d*" in the symbol $df(a,b)$ they concluded that the differential was some kind of slope or derivative. Indeed 19 of the 26 interviewed students showed this type of response. Tania is one such student. In problem 1 she showed no difficulty finding the slopes in the x and y directions of a given plane (m_x and m_y), finding the total vertical change (Δz) for Δx=4 and Δy=5, and writing the equation of the plane. Further, in problem 2 she correctly found the sign of the requested partial derivative by identifying it with the slope of a tangent line she drew on the given graph.

Tania: (In problem 3) What does it mean by the differential of f ? Is that a slope?

Interviewer: The differential of f at a point. That was defined in class. What is the meaning of the differential of a function?

Tania: That was the slope at this point, isn't it?

Interviewer: No.

Tania: Ok, that would be, the vertical change.

Interviewer: OK

Tania: The point (1,2) is this point, but to look for a vertical change I need two points, to be able to look for a z.

Note that Tania might be thinking of vertical change along the graph of the function (hence her need for two points). If this was the case, then she was not looking at the information about the function that may be obtained from the given tangent plane. This could indicate that she had not constructed the relation between the differential and the tangent plane. If she was thinking about vertical change along a plane, then she would seem not to have constructed a process of total vertical change on a plane, $\Delta z=\Delta z_x+\Delta z_y=m_x\Delta x+m_y\Delta y$, suggesting that she succeeded in problem 1 by applying Actions (using memorized formulas). In any case, she did not do treatment Actions on the tangent plane Object in order to obtain a formula for the differential, nor was she able to do a conversion Action to relate the analytical and graphic representations of the tangent plane in order to obtain the differential from the graph of the tangent plane, as conjectured in the GD. Considering that she might be showing difficulty thinking of *dx* and *dy* as independent variables in the sense discussed in the introduction, the interviewer asked:

Interviewer: If I tell you that the differential has to do with the variables *dx* and *dy* as independent variables, do you remember what is the differential?

Tania: Let's see, $df=m_x\,dx+m_y\,dy$.

Interviewer: Ok... and what is the differential then?

Tania: This would be *df* (she went on to correctly compute the total vertical change on the plane using $\Delta x=1$ and $\Delta y=1$ -rather than leaving *dx*, *dy* as independent variables- by calculating m_x and m_y).

Tania showed that she could identify a partial derivative with the slope of a tangent line in problem 2. Further, in problem 3 she could compute m_x and m_y from the given tangent plane. Hence, it seems she was able to obtain $f_x(1,2)$ and $f_y(1,2)$ from the graph of the tangent plane. However, she needed the interviewer's comment to help her remember a formula and to link it with the given plane, and did not consider *dx* and *dy* as independent variables. Overall, Tania seemed able to do some of the mental constructions described in the GD, however, her need for an explicit reminder and her apparent dependence on memorized formulas and procedures suggests that she had constructed an Action conception of the differential.

Ramon's performance on problem 1 suggested he had constructed a Process conception of total vertical change on a plane $\Delta z=m_x\Delta x+m_y\Delta y$. He was able to explain this formula in his own words, showing that he could imagine the geometric interpretation of the different components of the formula. Further, when obtaining the equation of the given plane in problem 1, he made clear reference to the notion of total vertical change on a plane.

However, when asked about the differential, he did not relate it to his conception of vertical change on a plane.

Ramón: I don't remember the formula for *df*.

Interviewer: And if I were to tell you that the differential of *f* gives the change in height along the tangent plane for horizontal changes of *dx* in the *x* direction and *dy* in the y direction?

Ramón: It is something like the formula for change, Δz… but I don't remember exactly how it was written… It was $df=m_xdx+m_ydy$… I don't know if it was something like this.

After correctly computing m_x and m_y he added:

Ramón: The Δx would be the d*x*, but at the point (1,2), that is, at $x=1$, $y=2$, around here… so it is this point here… I couldn't calculate it.

Interviewer: And if I tell you that *dx* and *dy* are independent variables?

Ramón: I don't have the change in x which is a very small number, no… I couldn't look for it there… the product of $m_x dx$ gives a change, vertical, but the *dx* alone only tells me it is a very small horizontal change in this figure…

It may also be observed that Ramón resists thinking of *dx* and *dy* as independent variables and instead thinks of them as "a very small horizontal change". On the other hand, Ramón gives some evidence of his ability to interpret the differential geometrically when he says "the product of $m_x dx$ gives a change, vertical..." Overall, Ramón's need of the interviewer's intervention suggests that he had constructed an Action conception of the differential. However, his Process conception of vertical change on a plane and his apparent ability to interpret geometrically the terms of the total differential tell us that perhaps he may have constructed an incipient Process conception of the differential.

Some students, like Karla, seemed to lose sight of the function once the tangent plane is given. Karla was able to quickly find and justify the sign of the partial and directional derivatives from the graph of the function given in problem 2. However, later when working with problem 3:

Interviewer: Could you tell me from the drawing of that plane there what is the meaning of directional derivative?

Karla: With this? [She pointed to the given plane and all she managed to do was darken the line segment on the plane where y=2].

When working with problem 3, Karla seemed unable to relate the tangent plane to the function when the function was not shown in the graph. She could not say anything about the partial derivatives, showing evidence of not considering the tangent plane as a local approximation of the function. Karla did not show any knowledge, or even recollection, about the total differential which led us to propose she had not even constructed an Action conception of this concept.

Results obtained from most students´ interviews were similar to those described here. Interview responses also evidenced that students need to explore the relationship between the differential and the notion of total vertical change on a plane, and that the GD should be revised to make this construction explicit, as well as the importance of recognizing the differential at a point as the total vertical change on the tangent plane as a function of the horizontal change (*dx*,*dy*). The information obtained from the interviews suggest the importance of making explicit the construction of *dx* and *dy* as independent variables, in the sense discussed in the introduction, when introducing the total differential, as well as the fact that an effective mental construction of the differential should include coordination between Processes on the function (e.g. partial derivatives, directional derivatives, and vertical change) and the same Processes on the tangent plane into a Process where the students can think of the tangent plane as a close approximation to the function locally. Results also demonstrate the need to make constructions related to treatments and conversions (Duval, 2006) on the tangent plane, and the dependence of the mental constructions of plane, tangent plane, and differential on a Process of vertical change on a plane.

Results suggest that the notion of vertical change on a plane can be institutionalized as a technique and a technology by using it to furnish geometric meaning to the notions of plane, tangent plane, total differential, directional derivatives (Martínez-Planell, Trigueros, and McGee, 2015) and to their symbolic representation, so that the students construct a meaningful understanding of differentiable functions. The introduction of tasks and techniques related to this technology can also be used to ascertain the adequacy of the moments of study and to promote students' learning.

DISCUSSION

The way to introduce the total differential concept is controversial. Many introductory textbooks take an approach that stresses symbolic computation using classical notation, disregarding the geometric insight that Tall (1992) suggested. Results of this study show that this approach is not appropriate. The analysis using the institutional tools from APOS-ATD dialogue show it constrains students' possibility to develop the necessary *praxeologies* given its restriction to action-techniques and the unbalance in the moments of study while results on students' constructions evidence that vertical change and the point-slopes equation for the tangent plane make a difference in students' understanding of the total differential. These results also underline that there is the implicit assumption in the textbook, that to understand this concept it suffices to recall the equation of the plane in R^3, along with its relation to the partial derivatives of a function, followed by numerical and symbolic computations to approximate values of a function in a neighbourhood of the point using the tangent plane. This assumption is not sustained by the results obtained in this study. Results show that the notion of differential appeared to be very difficult for students independently of the teaching method used.

In spite of the limited understanding shown by students during the interviews, results obtained can be considered an important contribution to the literature, since students' arguments through the interview and the comparison of responses from the three different groups of students help identify constructions, that seem to be fundamental in learning the concept of differential and that require the design of tasks needed for a complete and balanced *praxeology*. The necessary constructions are described in what follows.

The coordination of Processes of tangent plane to a surface and of two-variable function into a Process where students can recognize that the tangent plane on a point of the function can be considered as the local linear approximation of the function, both in an analytical and in a graphical representation, is important to understand that change in the surface can be approximated locally as change in the plane and thus in obtaining an analytical definition for the differential. It also plays an important role in allowing the construction of relations between the tangent plane and the differential to avoid the isolation of concepts students demonstrated in this study. For this, the construction of vertical change on a plane as a Process is necessary, in terms of the cognitive approach followed in this study. It seems that the normal equation for a plane is not enough for students to think of vertical change as an important characteristic of its description. An explicit relation between this equation and the point-slopes equation needs to be constructed through the coordination of the Processes of conversion involved in passing from one to the other.

Another construction needed is the Process involved in the recognition that the differential at a point $df(a,b)$ is a function of two variables dx and dy. Furthermore, comparison between the three groups involved in this experience demonstrated that the construction of the vertical change Process is necessary, but not sufficient; the coordination of this Process with the differential Process plays a fundamental role in surpassing the common idea found in this study that the differential is "some kind of derivative" or "a very small change".

Results obtained in general suggested that the GD needed to be refined and activity sets needed to be revised accordingly, taking into account results from the didactical analysis, to help students construct a process conception of the differential. A refined GD was designed as part of the contribution of this study. Both the new model, and re-designed activities need to be tested.

REFINED GENETIC DECOMPOSITION FOR THE DIFFERENTIAL OF TWO-VARIABLE FUNCTIONS

As a result of the analysis of student activity in the interviews the following constructions were added:

Action treatments and conversions in and between representations (Duval, 2006) are performed on the Process of total vertical change to encapsulate it into the Object vertical change on a plane.

Actions on the Object vertical change on a plane, of computing the vertical change from an initial point (x_0,y_0,z_0) on a plane to a final generic point (x,y,z) on the plane, and reflection on the relationship between the result of this Action and the point slopes formula for the plane, $z–z_0=m_x(x – x_0)+m_y(y – y_0)$, may be interiorized into a Process conception of plane.

A Process of location of base point and direction vectors is coordinated with a Process of two-variable function and a Process of derivative of function of one variable to form Processes that enable students to think about, imagine, and physically or geometrically represent curves on the graph passing through the base point in different directions, tangent lines to these curves passing through the base point in the given vector directions, and the tangent plane to the graph of the function at the given base point. These Processes are coordinated into a new Process of tangent plane that enables students to recognize that the tangent plane at a point is the collection of all tangent lines at that point.

Actions of comparing the graph of a (differentiable) function with the graph of its tangent plane at increasingly smaller neighborhoods of a given base point (using technology), Actions of comparing vertical change in the graph of the function and the tangent plane in the same neighborhoods, and Actions of comparing partial derivatives of the function with the slopes in the *x* and *y* directions of the corresponding tangent plane are interiorized into Processes and coordinated into a new Process where the tangent plane can be considered as a local approximation of the surface in a proximity of the base point. This Process also allows recognition that the surface and the tangent plane will be practically the same in sufficiently small neighborhoods of the base point. Further Actions on the Process of tangent plane, for example, comparison with the normal equation of the same plane, can contribute to its encapsulation into the Object tangent plane.

Actions to express the point-slopes equation for the tangent plane to a surface represented by $z = f(x,y)$ and total vertical change on the tangent plane, $\Delta z=m_x\Delta x+m_y\Delta y$ at a given point $(a,b,f(a,b))$ as the total differential $[df(a,b)](dx,dy)=f_x(a,b)dx+f_y(a,b)dy$ are interiorized into a Process relating vertical change, the tangent plane, and total differential. Actions of evaluating the differential at a fixed point (a,b) for different values of *dx* and *dy* are interiorized into a Process that recognizes that given a function *f* and a point (a,b), the differential $df(a,b)$ is the total vertical change on the tangent plane expressed as a function of the horizontal changes *dx* and *dy* (see Figure 2).

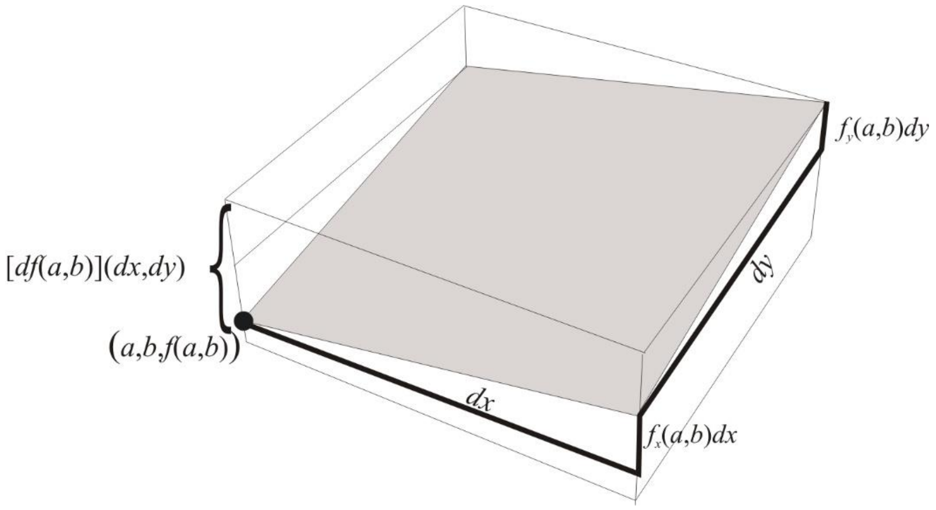

Figure 2: Total differential

Reflection on the Action of computing the differential at different points allows interiorization of the differential into a Process where the functional dependence of the differential on the starting point (a,b) is recognized. This process is coordinated with that of function so that the consideration of the differential as a two-variable function is made possible. When Actions need to be applied, for example, to find specific properties of the differential, it may be encapsulated into an Object.

CONCLUSIONS

Results from the comparison of control and experimental sections gave important information both, to understand why students showed specific difficulties, and to refine the genetic decomposition accordingly. We consider this information constitutes an important contribution of this paper to the literature and to the teaching of this concept. The use of the tools resulting from the APOS-ATD dialogue highlighted crucial aspects which provided a complementary vision of the role of teaching materials in students' learning.

The activities designed with the preliminary GD showed their potential in contributing to students learning: Students in the control section (1-Action conception, 8-lacked construction) and in the experimental section who had not used the activities (8-lacked construction) showed little or no understanding of the total differential while students in the experimental section who worked on the activities (1- Process conception, 5-Action conception, 3-lacked construction) showed better understanding. These differences underline their importance, and suggested constructions that were not included in the original GD. The didactical analysis of the activities indicated changes needed so they would contribute to the development of a complete praxeology for the differential. Comparison of results from the experimental section where only activities for the construction of plane and tangent plane were used show that constructing Processes of vertical change and tangent plane would not by themselves result in students' understanding of the differential.

The findings we describe in this paper enabled us to understand the nuances involved in the construction of the differential concept and, more importantly, why it is so difficult for students. They can also be of practical use for teaching the total differential.

CONFLICT OF INTEREST

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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