# HOW ARE GRAPHS OF TWO VARIABLE FUNCTIONS TAUGHT?

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*This is a study about how graphs of functions of two-variables are taught. We are interested in particular in the techniques introduced to draw and analyze these graphs. This continues previous work dedicated to students' understanding of topics of two-variable functions in multivariable calculus courses. The model of the "moments of study" from the Anthropological Theory of the Didactic (ATD) is used to analyze the didactical organization of the topic of interest in a popular calculus textbook, and in a typical classroom presentation. In so doing we obtain information about the institutional dependence of findings in previous studies.*

## Antecedents

Despite its importance, there are not many published articles in the mathematics education research literature that deal with the particularities of functions of two variables. The first published article we found that explicitly treats functions of two variables is by Yerushalmy (1997). In it he insisted on the importance of the interplay between different representations to generalize key aspects of these functions and to identify changes in what seemed to be fixed properties of each type of function or representation. Kabael (2009) studied the effect that using the "function machine" might have on student understanding of functions of two variables, and concluded that it had a positive impact in their learning. In other work, Montiel, Wilhelmi, Vidakovic, & Elstak (2009) considered student understanding of the relationship between rectangular, cylindrical, and spherical coordinates in a multivariable calculus course. They found that the focus on conversion among representation registers and on individual processes of objectification, conceptualization and meaning contributes to a coherent view of mathematical knowledge. Martínez-Planell and Trigueros (2009) investigated formal aspects of students' understanding of functions of two variables and identified many specific difficulties students have in the transition from one variable to two variable functions. Using APOS theory, they related these difficulties to specific coordinations that students need to construct among the set, one variable function, and $R^3$ schemata. Finally, in a study about geometric aspects of two variable functions, Trigueros and Martinez-Planell (2010) concluded that students' understanding can be related to the structure of their schema for $R^3$ and to their flexibility in the use of different representations. They gave evidence that the understanding of graphs of functions of two variables is not easy for students, that it can be related to the structure of students' schema for $R^3$, and in particular, that intersecting surfaces with planes, and predicting the result of this intersection, plays a fundamental role in understanding graphs of two variable functions and was particularly difficult for students.

The way students are taught, and the way mathematical topics are introduced in the textbooks used by students plays an important role on what they learn. In this study we analyze the way graphs of functions of two variables are presented in a widely used textbook, and in standard university classrooms. Our research questions for the part of the study we present here are:

- How is the topic "graphs of two-variable functions" introduced in a widely used textbook?

- How is this topic taught in a university class?
- Are conversions among representations favored?
- What relationships can be found between the above mentioned students' difficulties, the presentation methods used in the textbook, and the selected classrooms?

## Theoretical framework

In this article we incorporate Anthropological Theory of the Didactic (ATD) as a tool for the epistemological analysis of the textbook and classrooms. In this theory the mathematical activity and the activity of studying mathematics are considered parts of human activity in social institutions (Chevallard, 1997; Bosch and Chevallard, 1999). The theory considers that any human activity can be explained in terms of a system of *praxeologies*, or sets of practices which in the case of mathematical activity constitute the structure of what is called *mathematical organizations* (MO). Mathematical organizations always arise as response to a question or a set of questions. In a specific institution, one or several techniques are introduced to solve a task or a set of tasks. Tasks and the associated techniques, together form what is called the *practical block* of a praxeology. The existence of a technique inside an institution is justified by a technology, where the term "technology" is used in the sense of a discourse or explanation (*logos*) of a technique (*technè*). The technology is justified by a theory. A theory can also be a source of production of new tasks and techniques. Technology and theory constitute the *technological-theoretical block* of a praxeology. Thus a praxeology is a four-tuple ($T/\tau/\theta/\Theta$) (tasks, techniques, technologies, theories), consisting of a practical block, ($T/\tau$), the tasks and techniques, and a theoretical block, ($\theta/\Theta$), made up of the technological and theoretical discourse that explains and justifies the techniques used for the proposed tasks. Typically, a praxeology gives raise to new praxeologies as new problems are explored, techniques are generalized and different ones are introduced, the range of application of technologies expand, and the theoretical basis grows to encompass more general phenomena. This gives raise to mathematical organizations consisting of interrelated sets of praxeologies.

Within an educational institution a mathematical praxeology is constructed by a didactic process or a process of study of a MO. This process is described or organized by a model of six moments of study (Chevallard, 2007) which are: *first encounter* with the praxeology, *exploratory moment* to work with tasks so that techniques suitable for the tasks can emerge and be elaborated, *the technical work moment* to use and improve techniques, the *technological-theoretical moment* where the technological and theoretical discourse takes place, the *institutionalization moment* where the key elements of a praxeology are identified, leaving behind those that only serve a pedagogical purpose, and *evaluation moment* where student learning is assessed and the value of the praxeology is examined. It is important to clarify that the order of the moments is not fixed. It depends on the didactical organization in a given institution, but independently of the order it can be expected that there will be instances where the class will be involved in activities proper to each of the "moments".

This didactic model is used in this study to describe the mathematical organization related to functions of two variables presented both in the textbook used by students and in the work done in class. This description will be helpful when looking for relations between results of the analysis and students' difficulties and constructions found in the literature. The use of results of this endeavor can be helpful in the design and analysis of activities of a didactic sequence in terms of their institutional suitability with the purpose to help students understanding the concept of two variable functions.

## Methodology

This study is related to a project being conducted by the authors in two universities in different countries. A textbook was selected by the researchers to be analyzed considering that it is used in both universities involved in the study, and widely used in other universities. Two researchers independently reviewed the text in terms of the theoretical framework and negotiated their findings until agreement was reached. One of the researcher observed several classes where the topic was introduced by different teachers, took notes about the way functions of two variables was taught by different teachers and interviewed some of them. After transcription of observations and interviews, data was analyzed again by two researchers and results negotiated between them. Results obtained were compared with those found in previous studies and with the genetic decomposition suggested in Trigueros and Martinez-Planell (2010) in terms of the constructions this model proposes for the learning of the topic in order to look for possible relations of constructions found to be made by students and the way functions of two variable are introduced in the text and in classrooms.

In this study we concentrate on results related to graphs of two variable functions. For this purpose, we take into account the analysis of the selected textbook and the classroom observations and interview from one of the teachers who represent the way most of the observed teachers taught this topic to their students.

## Analysis of a textbook based on the ATD moments of study

Graphs of functions of two variables are introduced in courses of multivariable calculus to help students construct a richer mental model to reason about these functions and to illustrate the important concepts of the differential and integral calculus of this type of functions.

In the selected text, Calculus: early transcendentals by James Stewart, $6^{th}$ edition (2006), multivariable functions are introduced in the text in chapter 14 devoted to partial derivatives. However, fundamental planes (that is, planes parallel to the coordinate planes), considered as a prerequisite to understand these functions, are met before, in Section 12.1, where the three-dimensional coordinate system is introduced. This is done through the introduction of the tasks of graphing the planes $z=3$ and $y=5$. This can be considered a *moment of the first encounter* with graphs of two variable functions. These tasks remain isolated since they are not afforded any special role in relation to their use in understanding other subsets of $R^3$ (by forming intersections, for example), and are not met again until the exercises at the end of the section. Its only connection with a mathematical organization is their appearance in a section devoted to three-dimensional space. There is another *moment of the first encounter* for the topic of graphing functions of two variables in Section 12.6 where the graphs of quadric surfaces are shown for the first time. Even though quadric surfaces are not always graphs of functions of two variables, the task of graphing an ellipsoid with equation $x^2 + y^2/9 + z^2/4 = 1$ is introduced, and the technique to do the task explained: *"By substituting $z = 0$, we find that the trace in the xy-plane is $x^2 + y^2/9 = 1$, which we recognize as an equation of an ellipse"*, and immediately generalized to a families of traces: "*In general, the horizontal trace in the plane $z = k$ is $x^2 + y^2/9 = 1 - k^2/4$ …*" . We found another example of a *moment of first encounter* in Section 14.1 where after defining functions of two variables, the text presents the first examples of graphs of functions. A linear function and the top half of a sphere are graphed by recognizing their types of equations (no use of traces), then a graph is generated by a computer without using or mentioning traces, and finally an elliptic paraboloid is presented, one that had been presented before, by making reference to its prior appearance.

We can observe an absence of questions generating the need for graphing functions, nor the importance of understanding how to graph them. These facts, together with the isolation of the tasks presented, are not conductive to a mathematical activity where techniques arise in a productive way in terms of students' learning.

In the *exploratory moment* in Section 12.6, the tasks of graphing five quadric surfaces are introduced, perhaps not as an end in and of itself, but as a means to establish the technique of traces that expectedly is to be further developed with the continued exploration of other tasks. Although at first sight this number of might seem quantitatively adequate, the examples and their accompanying explanation require students from the outset to recognize and place in space a family of curves, a task that has not been introduced before. The tasks in this exploratory moment are not adequate to prepare students to use traces as a technique to draw the graphs of two variable functions. The exploration continues in Section 14.1, where the text mentions the use of traces in computer generated graphs and shows four such graphs with hardly any comment. Here, the technique of using traces to draw graphs of function is related to tasks done with technology, but the way it is presented makes it difficult for students to interpret how the technique works in this mathematical organization.

As we can see the textbook's moment of exploration does not present students with opportunities to encounter relevant tasks which can help them make sense of what the traces shown are about, they also are not given real opportunities to explore the tasks in order to find regularities or properties which can help them make sense of the technique that is being introduced.

To examine the *moment of practice of the technique* we found that in Section 12.1 there are some exercises that make direct or indirect use of fundamental planes to describe regions in three-dimensional space. Only a few of them are assigned exercises in the syllabus of both courses. Further and more importantly, these exercises do not provide the opportunity of exploring the result of intersecting fundamental planes with subsets of $R^3$ and to relate these intersections with the technique of traces introduced before. Then, in Section 12.6, relatively few exercises at the end of the section require using sections to produce the graph of a surface, and only six of them are assigned in the courses' syllabus. In Section 14.1 we found some exercises that require using cross-sections to draw the graph of a two variable function, but most of them are presented in terms of matching problems. This kind of problems would be useful to exercise the technique of drawing graphs of surfaces if they required students to justify their selections using cross-sections. As they are presented, students tend to attempt using other strategies, frequently without success, and hence these tasks do not really give students the opportunity to practice the technique introduced. Exploratory moments are not integrated and systematic; the text has hardly any task where interpretation or justification of the technique is needed. It introduces other techniques, as the recognition of the algebraic form of quadric surfaces, but does not relate it with the use of traces.

We can say that the text does not provide enough opportunities for the students to work on the practical block of the praxeology and no ground is set to develop consistently the theoretical part of the praxeology related to graphing functions and converting flexibly among different representation registers.

As discussed in Chevallard (2007), the *moment of development of technology and theory* is closely interrelated with each of the other moments of study. This is clearly seen to be the case in this topic. The technology of using traces or cross-sections to draw the graph of a function of two variables is introduced in the moment of first encounter and developed with scant opportunities to do task explorations, as discussed above. In the book, there is not an explicit discussion of the fact that substituting a number for a variable in an equation with three variables corresponds to intersecting a fundamental plane with the graph of the equation. Hence, there is hardly an explanation about cross-sections, projections, and contours. The examples discussed in Section 12.6 assume that students can readily recognize families of curves and place them in space; the reader is left to make sense by him or herself, or resort to a memorized table of surfaces and formulas to answer questions about graphs of two variable functions.

From the point of view of ATD, the lack of a technology to make sense of the technique introduced leaves the mathematical organization ungrounded. Students may not understand why the graph of a function of two variables is important, why it is a surface and how to make sense of even computer generated graphs. The mathematical organization constructed consists of isolated ideas. It does not have coherence.

Consider now the *moment of institutionalization.* From the point of view of the two educational institutions, as documented in the course syllabus, this is a topic to be mastered. Differential and integral calculus of functions of two variables is to a large extent based on reducing problems of functions of two variables into problems of functions of one variable by holding a variable fixed and analyzing the resulting one-variable function. The textbook can be considered as the reference for the institutionalization of those elements of the mathematical organization that are considered as mathematical objects. However, as discussed before, the way graphs of functions are presented does not provide any ground for students to be aware of the importance of this idea. Moreover, quadric surfaces are not used again in any substantial way after their introduction in Section 12.6, their importance and their role is never clarified. They remain isolated from the rest of the mathematical organization being presented in multivariable calculus. Even if it appears that the intention was to use them, as a means to introduce sections in graphing surfaces, this goal is not achieved as we have seen. They are not reinforced in Chapter 14 where functions of two and three variables are introduced. The *moment of institutionalization* of this topic is not clear.

The *moment of evaluation* of what is learned by students is done by the instructors teaching the course. Although the textbook includes some review questions and a test, for the most part they are not related with graphing functions of two variables, and they are not used by the teachers. If we review the courses' syllabus we find that students are evaluated using three or four partial examinations and a final examination. In one of the universities the topic of graphing surfaces comes late in the semester of Calculus II, and is not included in a partial examination, so it accounts for no more than 3% of the final grade but in Calculus III the topic is evaluated more thoroughly. At the other university the topic is covered at the beginning of Calculus III and is evaluated in a partial examination. Despite the fact that the analysis of graphs of two variable functions by considering its restriction to fundamental planes is pervasive throughout the study of calculus of two variables, its introduction through the textbook and in evaluations, does not stress that importance. Students get the impression that this is not an important topic and may not pay attention to it. There is not a clear presence of the *moment of evaluation* of the technique itself in the textbook.

## Analysis of the teaching of the graphs of two variable functions

As can be expected, professors in their class complement the information given in textbooks. We now use the moments of study to analyze the data we obtained from observing the classes of a particular professor during the two weeks he devoted to the teaching of functions of two variables, as an illustration of how the teachers work in class, since we found out many similarities in the work of all the teachers observed, even though they teach in different countries. Again, we focus here on their graphs.

After a brief discussion of one variable functions where the teacher emphasizes graphs of these functions, in particular linear and quadratic functions, he introduces functions of two variables as an extension of functions of one variable and asks students how they would graph this type of function and what kind of geometrical object the graph would be. After discussion with the whole group the teacher leads the students to the fact that the graph should be a surface. He introduces the equation of a plane in $R^3$ and its relationship to the graph of the plane, using fundamental planes in the analysis. Students are given a few tasks where they have to draw planes parallel to the coordinate planes and to find their equation. The *moment of the first encounter* with two variable functions is presented in terms of the relationship between functions of one and two variables and the question of how the graph would look like. Fundamental planes are introduced but there is no justification of why they are needed.

Later on, the class as a whole discusses quadric surfaces; the tasks for discussion consist in finding the conic curves that combined give raise to different surfaces. Then tasks are given to the students where they have to find intersections of a surface with fundamental planes in order to graph a given function and find its domain and range. Thus the *moment of task exploration* consists of a series of tasks designed for the students to be aware of the difficulties involved in graphing functions of two variables and how they can use some strategies, such as using fundamental planes, including intersections with the coordinate planes to help them doing so. Graphs of functions are also linked with other properties of functions of two variables. An example of a task is: draw the graph of the function $z=2-y^2$ by using different planes of the form $x=k$. It is interesting to note that when students use the planes, the equation remains the same, so they have to discuss the difference between the graph of the function and the intersection curves. The teacher then gives homework where students can use computer generated graphs to relate the equation of quadric surfaces with their graphs. In the moment of task exploration students work with graphing functions of two variables using fundamental planes. The technique is introduced as a means to facilitate graphing the functions and it is also linked to the domain and range of the given functions. We consider that up to this point, the teacher is constructing the practical part of the praxeology. We noticed, however that this moment is not expanded to the homework; it goes back to graphs and equations of quadric surfaces without connection to the introduced tasks.

The *moment of practice of the technique* is quite restricted. After introducing the technique the teacher goes back to the definition of function of two variables and gives students some tasks where they have to find domain, range and graph of several functions. Students can either use what they have reviewed about quadric surfaces and their graphs, or fundamental planes. Most of the tasks are related to quadric surfaces. Students do not have enough opportunities to work with the technique, although it seems clear to them that the use of fundamental planes is related to finding the graph of the function.

While working with the technique there is some explicit discussion of the fact that substituting a number for a variable in an equation with three variables corresponds to intersecting a fundamental plane with the graph of the equation, and some explanation about contours and projections. However, since the number of tasks worked by the students is reduced, in our opinion the *moment of the development of the technology* is not well developed, this fact may leave students with a superficial idea of the importance of the technique, although they may be able to use it with simple functions. The technological-theoretical block of the praxeology is not well developed in this class.

The professor works with other examples when finding graphs of functions, and each time he makes clear that fundamental planes, contours and projections are useful to understand how the function "behaves". This repeated emphasis can be considered as the *moment of institutionalization* of the topic we are concerned with.

In the first partial exam the teacher asks students to draw the graph of a simple function using fundamental planes. He also asks a question where students have to find the intersection of a fundamental plane with a given surface and they have to graph the resulting curve in the plane. This can be considered *the evaluation moment*.

## Discussion and conclusions

Results of the previous analysis show that even though curriculum and teachers underline the importance of the graphs of functions of two variables in relation to their study, the components of the praxeology are presented in an isolated and incomplete form in the textbook used in the analyzed classrooms: its *practical block* is presented in a very superficial way, there are very few opportunities for students to do *tasks* where they can establish relationships among different representations, and the *technological-theoretical block* is absent. This presentation is bound to be ineffective, in terms of learning. Its importance is also not communicated to the reader. We can also see that although professors, as demonstrated by the one described here, present this praxeology with more elements, including more work on *tasks* relating representations, in terms of the *moments of study*, this presentation is also incomplete. It consists of quite isolated elements, since the *technique* introduced is not practiced enough and is not clearly related to a *technological* discourse.

In previous research Trigueros and Martínez-Planell (2009, 2010) investigated the relationship between students' notion of subsets of Cartesian three-dimensional space, and their understanding of graphs of two-variable functions with students who had taken courses as the one described here and used the textbook as a reference, using APOS theory and Duval's theory of representations as a theoretical framework. Results from these studies showed that understanding of two variable functions is not easy for students and can be related to the structure of their schema for $R^3$ and to their flexibility in the use of different representations. In particular, it was shown that students who had completed a multivariable calculus course present many difficulties with constructions involving fundamental planes and surfaces in Cartesian space. Most of the interviewed students had difficulty relating information about two variable functions in different representation registers, did not use fundamental planes to draw graphs, showed confusion between surfaces, curves and solids in space and only one of them was found to be able to intersect surfaces with planes and predict the result of this action. Results also demonstrated that most students have many difficulties understanding functions of two variables, their domain and range, and that the generalization of understanding of one variable functions to two-variable functions, in particular in the case of graphical representation, is not direct. The study showed, for example, that many students do not readily convert the action "substituting $z = 0$", which is a *task,* into that of intersecting a fundamental plane with the surface, so they are not ready to consider families of traces. They need more opportunities to work on tasks that help them interiorize actions into processes, and thus build the necessary *techniques*.

Results of the present study, together with those of the previous ones, show a more complete and detailed picture of observed phenomena. It is not surprising that students do not learn how to use traces to draw graphs of two variable functions. It is also comprehensible why they have difficulties to distinguish different subsets of Cartesian space and do not clearly understand the importance of being able to analyze graphs of these functions. The results obtained from the analysis of the *moments of study* of the mathematical activity associated with this topic, demonstrate that not enough opportunities are given to the students to master the *techniques* and *technologies* needed to analyze and use graphs of two variable functions. So, an effort needs to be done to balance activity in class so than all the moments of study are present in the study of these functions and help students deepen their understanding. It is true that students can use technology to see the graphs of functions, but without the necessary tools to make sense of the information contained in them, they may not "see" what the teacher intends to show.

Taken together, results of both studies give information about what needs to be done. We have started some work in this direction. We have designed sets of activities aimed at helping students to construct a better understanding of functions of two variables. Activity sets are available on the internet at http://math.uprm.edu/~rafael/. The design of these activities follows the constructions modeled in the refined genetic decomposition described in Trigueros and Martinez-Planell (2010), and also takes into account the characteristics of the two universities where the studies took place.

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