Lecture 4: The Linear Time Selection

Selection Problem

Given a sequence of numbers $\langle a_1, \ldots, a_n \rangle$, and an integer i, $1 \leq i \leq n$, find the ith smallest element. When $i = \lceil n/2 \rceil$, it is called the median problem.

Example: Given $\langle 1, 8, 23, 10, 19, 33, 100 \rangle$, the 4th smallest element is 19.

Question: How do you solve this problem?

First Solution: Selection by sorting

Step 1: Sort the elements in ascending order with any algorithm of complexity $O(n \log n)$.

Step 2: Return the ith element of the sorted array.

The complexity of this solution is $O(n \log n)$.

Question: Can we do better?

Answer: YES, but we need to recall Partition(A, p, r) used in Quicksort!

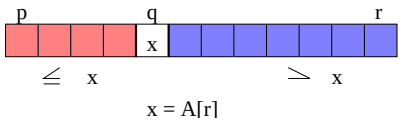
Second Solution : Linear running time in average

Recall of Partition(A, p, r)

Definition: Rearrange the array $A[p..r]$ into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$ such that

$$A[u] \leq A[q] < A[v]$$

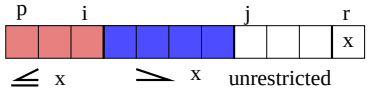
for any $p \leq u \leq q-1$ and $q+1 \leq v \leq r$.



(1) The original $A[r]$ is used as the pivot.
(2) It is a deterministic algorithm.
(3) The element for the qth position is found!

Note that this partition is different from the partition we used in COMP 171.

The Idea of Partition(A, p, r)

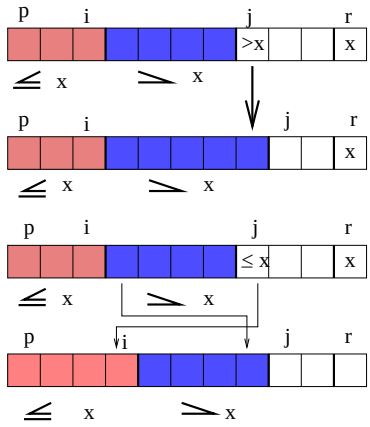


(1) Initially $(i, j) = (p-1, p)$.

(2) Increase j by 1 each time to find a place for $A[j]$. At the same time increase i when necessary.

(3) The procedure stops when $j = r$.

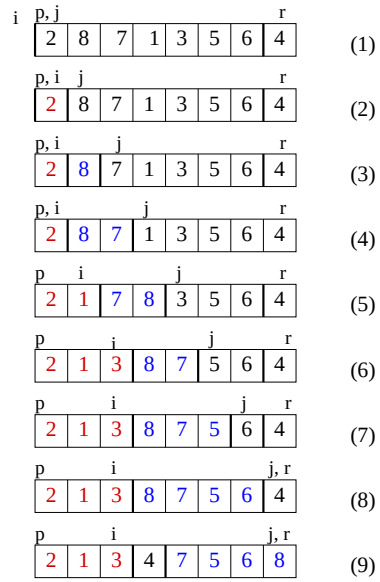
One Iteration of the Procedure Partition



(A) Only increase j by 1.

(B) $i \leftarrow i+1$. $A[i] \leftrightarrow A[j]$. $j \leftarrow j+1$.

The Operation of Partition(A, p, r): Example



The Partition(A, p, r) Algorithm

```
Partition(A, p, r)
{
 // A[r] is the pivot element
 x = A[r];
 i = p-1;
 for (j = p to r-1) {
  if (A[r] <= x) {
   i = i+1;
   exchange A[i] and A[j]
  }
 }

 // put pivot in position
 exchange A[i+1] and A[r]
 // q = i+1
 return i+1;
}
```


The Running Time of Partition(A, p, r)

comparison of array elements
assignment, addition, comparison of loop variables

Partition(A, p, r)**:**

$x = A[r]$	1
$i = p - 1$	1
for $j = p$ to $r - 1$	$2(r-p)$
if $A[j] \leq x$	$(r-p)$
$i = i + 1$	$\leq (r-p)$
exchange $A[i] \leftrightarrow A[j]$	$\leq 3(r-p)$
exchange $A[i+1] \leftrightarrow A[r]$	3
return $i + 1$	1

Total: $(r-p)$ and $\leq \{6(r-p)+6\}$
Running time is $\Theta(r-p)$, that is, linear in the length of the array $A[p..r]$.

Randomized-Partition(A, p, r)

The Idea: In the algorithm Partition(A, p, r), $A[r]$ is always used as the pivot x to partition the array $A[p..r]$.

In the algorithm Randomized-Partition(A, p, r), we randomly choose an j, $p \leq j \leq r$, and use $A[j]$ as pivot.

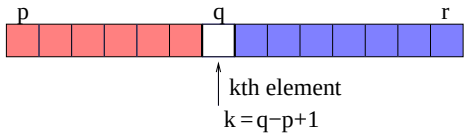
```
Randomized-Partition(A, p, r)
{
 j = random(p, r);
 exchange A[r] and A[j]
 Partition(A, p, r);
}
```

Remark: `random(p, r)` is a pseudorandom-number generator that returns a random number between p and r.

Randomized-Select(A, p, r, i), $1 \leq i \leq r - p + 1$

Problem: Select the ith smallest element in $A[p..r]$, where $1 \leq i \leq r - p + 1$.

Solution: Apply Randomized-Partition(A, p, r), getting



Case 1: $i = k$, pivot is the solution.

Case 2: $i < k$, the ith smallest element in $A[p..r]$ must be the ith smallest element in $A[p..q-1]$.

Case 3: $i > k$, the ith smallest element in $A[p..r]$ must be the $(i-k)$th smallest element in $A[q+1..r]$.

If necessary, recursively call the same procedure to the subarray.

Randomized-Select(A, p, r, i), $1 \leq i \leq r - p + 1$

if $p == r$
  return $A[p]$
$q =$ Randomized-Partition(A, p, r)
$k = q - p + 1$
if $i == k$ — the pivot is the answer
  return $A[q]$
else if $i < k$
  return Randomized-Partition$(A, p, q-1, i)$
else
  return Randomized-Partition$(A, q+1, r, i-k)$

Remark: To find the ith smallest element in $A[1..n]$, call Randomized-Select$(A, 1, n, i)$.

Running Time of Randomized-Select($A, 1, n, i$)

Let $T(n, i)$ be the average number of comparisons of array elements for $1 \leq i \leq n$.

Then $T(1, 1) = 0$ and for $n > 1$ we get

$$\begin{aligned} T(n,i) = {} & (n-1) && \text{initial partition} \\ & + \tfrac{1}{n}\{\textstyle\sum_{k=1}^{i-1} T(n-k, i-k) && \text{recursion, } k < i \\ & \quad + \textstyle\sum_{k=i+1}^{n} T(k-1, i)\} && \text{recursion, } k > i \end{aligned}$$

We will prove by induction on n that

$$T(n,i) < 4\,n$$

for all n and i.

Proof that $T(n,i) < 4\,n$

Induction basis: $T(1,1) = 0 < 4 \cdot 1$.
Induction step: Assume that $T(m,j) < 4\,m$ for all $m < n$ and $1 \le j \le m$. Then

$$\begin{aligned} & T(n,i) \\ &= n-1+\frac{1}{n}\{\sum_{k=1}^{i-1} T(n-k,i-k) + \sum_{k=i+1}^{n} T(k-1,i)\} \\ &< n-1+\frac{1}{n}\{\sum_{k=1}^{i-1} 4(n-k) + \sum_{k=i+1}^{n} 4(k-1)\} \\ &= n-1+\frac{1}{n}\{4n(i-1) - 4\frac{i(i-1)}{2} + 4\frac{n(n-1)}{2} - 4\frac{i(i-1)}{2}\} \\ &= n-1+\frac{1}{n}\{2n^2 - 6n + (4n+4)i - 4i^2\}. \end{aligned}$$

Proof that $T(n,i) < 4\,n$

$$T(n,i) < n-1+\frac{1}{n}f(i),$$

where

$$f(x) = 2n^2 - 6n + (4n+4)x - 4x^2.$$

$$f'(x) = (4n+4) - 8x = 0$$

$$f''(x) = -8 < 0$$

for $x = (n+1)/2$. Hence

$$f(x) \le f((n+1)/2) = 3n^2 - 4n + 1$$

for all x. Therefore

$$T(n,i) \le n - 1 + 3n - 4 + \frac{1}{n} < 4n.$$

Running Time of Randomized-Select($A, 1, n, i$)

We proved that $T(n,i) < 4n$. Since $T(n,i) \ge n-1$, we have in particular that

$$T(n,i) = \Theta(n).$$

Randomized-Quicksort Algorithm

We make use of the Randomized-Partition idea to develop a new version of quicksort.

```
Randomized-Quicksort(A, p, r)
{
 if (p < r) {
  q = Randomized-Partition(A, p, r);
  Randomized-Quicksort(A, p, q-1);
  Randomized-Quicksort(A, q+1, r);
 }
}
```

Does it run faster than the original version of quicksort?

Running Time of the Randomized-Quicksort

Results :

Worst Case: $T(n) = \Theta(n^2)$.
Average Case: $T(n) = O(n \log n)$.

Clearly, the worst case is still $\Theta(n^2)$, what about the average case?

Average running time of Randomized-Quicksort

Key observations:

- The running time of (randomized) quicksort is dominated by the time spent in (randomized) partition. In the partition procedure, the time is dominated by the *number of key comparisons*.
- When a pivot is selected, the pivot is compared with every other elements, then the elements are partitioned into two parts accordingly.
- Elements in different partition are NEVER compared with each other in *all* operations.

Tricks: We find the *expected* number of comparisons for **all** randomized-partition calls.

Average running time of Randomized-Quicksort

Let A be the input array which is a permutation of the n distinct elements $z_1 < z_2 < \ldots < z_n$.

Let X be the total number of comparisons performed in ALL calls to randomized-partition. Let X_{ij} be the number of comparisons between z_i and z_j, observe that X_{ij} can only be 0 or 1. Our goal is to compute the expected value of X, i.e.,

$$\begin{aligned} E[X] &= E[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [Pr\{z_i \text{ is compared to } z_j\} \times 1 \\ &\quad + Pr\{z_i \text{ is not compared to } z_j\} \times 0] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr\{z_i \text{ is compared to } z_j\} \end{aligned}$$

Average running time of Randomized-Quicksort

It remains to show how to find $Pr\{z_i \text{ is compared to } z_j\}$.

For $1 \le i \le j \le n$, let $Z_{ij} = \{z_i, z_{i+1}, \ldots, z_j\}$ (remember $z_i < z_{i+1} < \ldots < z_j$).

Key observations:

- If z_i or z_j is selected as a pivot BEFORE any elements in $\{z_{i+1}, z_{i+2}, \ldots, z_{j-1}\}$, z_i and z_j will be compared.
- Conversely, if any element in Z_{ij} other then z_i or z_j is selected as a pivot before z_i and z_j, z_i and z_j will be placed in DIFFERENT partitions, and hence they will NOT compare with each other in ALL randomized-partition calls.
- ANY element other than the elements in Z_{ij} has no effect to $Pr\{z_i \text{ is compared to } z_j\}$.

Average running time of Randomized-Quicksort

It remains to find the probability that z_i or z_j is the first pivot chosen from Z_{ij}.

$$
\begin{aligned}
&Pr\{z_i \text{ is compared to } z_j\} \\
&= Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\} \\
&= Pr\{z_i \text{ is the first pivot chosen from } Z_{ij}\} \\
&\quad + Pr\{z_j \text{ is the first pivot chosen from } Z_{ij}\} \\
&= \frac{1}{j-i+1} + \frac{1}{j-i+1} \\
&= \frac{2}{j-i+1}
\end{aligned}
$$

Average running time of Randomized-Quicksort

Putting everything together, we have

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} \\
&< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \\
&= \sum_{i=1}^{n-1} O(\lg n) \\
&= O(n \lg n)
\end{aligned}
$$

Hence, the expected number of comparisons is $O(n \lg n)$, which is the average running time of Randomized-Quicksort.