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Further analysis of the number of spanning trees in circulant graphs☆

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Abstract

Let $1 \leqslant s_1 < s_2 < \cdots < s_k \leqslant \lfloor n/2 \rfloor$ be given integers. An undirected even-valent circulant graph, $C_n^{s_1,s_2,\ldots,s_k}$, has n vertices $0, 1, 2, \ldots, n-1$, and for each s_i $(1 \leqslant i \leqslant k)$ and j $(0 \leqslant j \leqslant n-1)$ there is an edge between j and $j + s_i \pmod n$. Let $T(C_n^{s_1,s_2,\ldots,s_k})$ stand for the number of spanning trees of $C_n^{s_1,s_2,\ldots,s_k}$. For this special class of graphs, a general and most recent result, which is obtained in [Y.P. Zhang, X. Yong, M. Golin, [The number of spanning trees in circulant graphs, Discrete Math. 223 (2000) 337–350]], is that $T(C_n^{s_1,s_2,\ldots,s_k}) = na_n^2$ where a_n satisfies a linear recurrence relation of order 2^{s_k-1}. And, most recently, for odd-valent circulant graphs, a nice investigation on the number a_n is [X. Chen, Q. Lin, F. Zhang, The number of spanning trees in odd-valent circulant graphs, Discrete Math. 282 (2004) 69–79].

In this paper, we explore further properties of the numbers a_n from their combinatorial structures. Comparing with the previous work, the differences are that (1) in finding the coefficients of recurrence formulas for a_n, we avoid solving a system of linear equations with exponential size, but instead, we give explicit formulas; (2) we find the asymptotic functions and therefore we 'answer' the open problem posed in the conclusion of [Y.P. Zhang, X. Yong, M. Golin, The number of spanning trees in circulant graphs, Discrete Math. 223 (2000) 337–350]. As examples, we describe our technique and the asymptotics of the numbers.

Keywords: Circulant graphs; Number of spanning trees; Recurrence relations; Asymptotics

1. Introduction

All graphs here will be undirected if not otherwise stated, and can have multiple edges and self-loops. A *spanning tree* in a graph G is a tree having the same vertex set as G. The study of the number of spanning trees in a graph has a long history and has been very active because finding the number is important: (1) in estimating the reliability of a network; (2) in analyzing energy of masers in investigating the possible particle transitions; (3) in designing electrical circuits etc. [3,5,8,10]. A classic result on this problem is the *matrix tree theorem* [11] which expresses the number of spanning trees $T(G)$ in terms of the determinant of a matrix that can be easily constructed from G's adjacency matrix. However, counting the numbers by directly calculating this determinant is not acceptable for large graphs. For this

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reason people have developed techniques to get around the difficulties (see, for example, [4] and references therein) and have paid more attention to deriving explicit and simple formulas for special classes of graphs, see [2,5,8,14,15] for recent work.

Let $s_1, s_2, \ldots, s_k$ be fixed positive integers. In our considerations, without loss of generality we may assume that $1 \leqslant s_1 < s_2 < \cdots < s_k \leqslant \lfloor n/2 \rfloor$. A circulant graph $C_n^{s_1,s_2,\ldots,s_k}$ has n vertices labelled $0, 1, 2, \ldots, n-1$, with each vertex i $(0 \leqslant i \leqslant n-1)$ adjacent to vertices $i+s_1, i+s_2, \ldots, i+s_k \pmod n$. A directed circulant graph, $\vec{C}_n^{s_1,s_2,\ldots,s_k}$, is a digraph on n vertices $0, 1, 2, \ldots, n-1$; for each vertex i $(0 \leqslant i \leqslant n-1)$, there are k arcs from i to vertices $i+s_1, i+s_2, \ldots, i+s_k \pmod n$. Note that $C_n^{s_1,s_2,\ldots,s_k}$ is a $2k$-regular graph, but $\vec{C}_n^{s_1,s_2,\ldots,s_k}$ is k-regular. As usual, we use $T(C_n^{s_1,s_2,\ldots,s_k})$ to signify the number of spanning trees of circulant graph $C_n^{s_1,s_2,\ldots,s_k}$ and $T(\vec{C}_n^{s_1,s_2,\ldots,s_k})$the number of spanning trees in directed circulant graph $\vec{C}_n^{s_1,s_2,\ldots,s_k}$.

During the past decades, for some special s_j's the recurrence formulas for $T(C_n^{s_1,s_2,\ldots,s_k})$ have been studied extensively. Starting from the different proofs [12,2,14] of the conjecture $T(C_n^{1,2}) = nF_n^2$, where F_n the *Fibonacci* numbers, of Bedrosian [3] (which was also conjectured by Boesch and Wang [6] without the knowledge of [11]), the formulas for $T(C_n^{1,3})$, $T(C_n^{1,4})$ and more general result have recently been obtained in [5,1,14,17,16], where the most general formula is obtained in [16], which proves that $T(C_n^{s_1,s_2,\ldots,s_k}) = na_n^2$ where a_n satisfies a linear recurrence relation of order 2^{s_k-1}.

Due to the reason that all the previous papers have to solve a system of linear equations to find the coefficients of the recurrence relation of a_n, we continue the work and focus on exploring further properties of the numbers a_n from their combinatorial structures. With the properties we will give, we do not have to solve such a system of linear equations to find the recurrence relations of a_n, but instead we give explicit and simple formulas in finding the coefficients. In this paper, we also show that the asymptotics of $T(C_{n+1}^{s_1,s_2,\ldots,s_k})/T(C_n^{s_1,s_2,\ldots,s_k})$ depend continuously on these s_j and k. This ‘answers’ the problem posed in the Conclusion of [17] where the problem asked is to characterize if the asymptotics of (when n tends to ∞)

$$T(C_{n+1}^{s_1,s_2,\ldots,s_k})/T(C_n^{s_1,s_2,\ldots,s_k})$$

depends only on some of these s_j and k, similar to the result obtained in [13] for directed circulant graphs (when n tends to ∞, $T(\vec{C}_{n+1}^{s_1,s_2,\ldots,s_k})/T(\vec{C}_n^{s_1,s_2,\ldots,s_k}) \sim k$, the degree of the vertices). As examples, we describe our technique and the asymptotic properties for the numbers.

2. Basic results

In this section, we will consider show that the number of spanning trees satisfies either a reciprocal or an anti-reciprocal recurrence relation in n. Following lemma is known.

Lemma 1 (*Chen et al. [7]*). *For any integer* $1 \leqslant s_1 < s_2 < \cdots < s_k \leqslant \lfloor n/2 \rfloor$,

$$T(C_n^{s_1,s_2,\ldots,s_k}) = \frac{1}{n}\prod_{j=1}^{n-1}(2k - \varepsilon^{-s_1 j} - \cdots - \varepsilon^{-s_k j} - \varepsilon^{s_1 j} - \cdots - \varepsilon^{s_k j}),$$

where ε^{-j} *is the conjugate of* ε^j, $\varepsilon = e^{2\pi i/n}$.

For convenience, let

$$g_{s_1,s_2,\ldots,s_k}(x) = 2k - x^{-s_1} - x^{-s_2} - \cdots - x^{-s_k} - x^{s_1} - x^{s_2} - \cdots - x^{s_k}, \tag{1}$$

$$f_{s_1,s_2,\ldots,s_k}(x) = c_0 x^{2s_k-2} + c_1 x^{2s_k-3} + \cdots + c_{s_k-1}x^{s_k-1} + \cdots + c_1 x + c_0, \tag{2}$$

where $f_{s_1,s_2,\ldots,s_k}(x)$ is a real reciprocal polynomial of degree $2s_k - 2$, and

$$c_{s_k-i} = \sum_{j=1}^{k}(s_j - i + 1)^+, \quad i = 1, 2, \ldots, s_k, \tag{3}$$

and

$$u^+ = \begin{cases} u, & u > 0, \\ 0, & u \leqslant 0. \end{cases}$$

Then

$$f_{s_1,s_2,\ldots,s_k}(x) = -\frac{x^{s_k}}{(x-1)^2} g_{s_1,s_2,\ldots,s_k}(x). \tag{4}$$

Since

$$x^{2n} + 2x^{2n-1} + \cdots + (n+1)x^n + \cdots + 2x + 1 = (1 + x + x^2 + \cdots + x^n)^2, \tag{5}$$

Eq. (4) can be seen directly from (5).

Note that the coefficients c_{s_k-i} of $f_{s_1,s_2,\ldots,s_k}(x)$ are uniquely determined by the numbers $s_1, s_2, \ldots, s_k, k$. Therefore the roots of $f_{s_1,s_2,\ldots,s_k}(x)$ depend continuously on these s_j, and k. Combining (3) and Lemma 2 in [17], we have the following Lemma 2.

Lemma 2.

$$T(C_n^{s_1,s_2,\ldots,s_k}) = n\frac{(-1)^{(s_k-1)(n-1)}}{f_{s_1,s_2,\ldots,s_k}(1)} |I - M^n_{s_1,s_2,\ldots,s_k}|, \tag{6}$$

where

$$M_{s_1,s_2,\ldots,s_k} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -c_{s_k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -c_2 \\ 0 & 0 & \cdots & 1 & -c_1 \end{pmatrix}$$

is the companion matrix of $f_{s_1,s_2,\ldots,s_k}(x)$ *and* $|X|$ *represents the determinant of matrix* X.

This lemma will be used for calculating the initial numbers of a_n.

Lemma 3. *For any integer* $1 \leqslant s_1 < s_2 < \cdots < s_k \leqslant \lfloor n/2 \rfloor$, *let*

$$T(C_n^{s_1,s_2,\ldots,s_k}) = n a_n^2,$$

then

$$a_n = \frac{1}{\sqrt{f_{s_1,s_2,\ldots,s_k}(1)}} \sum_{i=1}^{2^{s_k-2}} (-1)^i \begin{cases} r_i^n + r_i^{-n}, & s_k \text{ is odd}, \\ r_i^n - (-r_i)^{-n}, & s_k \text{ is even}. \end{cases}$$

Proof. From (1), (2) and the above discussions,

$$\begin{aligned} T(C_n^{s_1,s_2,\ldots,s_k}) &= \frac{1}{n} \prod_{j=1}^{n-1} \varepsilon^{-s_k j} (\varepsilon^j - 1)^2 f_{s_1,s_2,\ldots,s_k}(\varepsilon^j) \\ &= n(-1)^{(s_k-1)(n-1)} \prod_{j=1}^{n-1} f_{s_1,s_2,\ldots,s_k}(\varepsilon^j). \end{aligned}$$

Thus

$$a_n^2 = (-1)^{(s_k-1)(n-1)} \prod_{j=1}^{n-1} f_{s_1,s_2,\ldots,s_k}(\varepsilon^j). \tag{7}$$

Note that if α_i is a real root of $f_{s_1,s_2,\ldots,s_k}(x)$ then $1/\alpha_i$ is also a root of $f_{s_1,s_2,\ldots,s_k}(x)$; if β_j is a complex root of $f_{s_1,s_2,\ldots,s_k}(x)$ then $1/\beta_j$, $\bar{\beta}_j$ and $1/\bar{\beta}_j$, where $\bar{\beta}_j$ is the conjugate of β_j, are also roots of $f_{s_1,s_2,\ldots,s_k}(x)$ because $f_{s_1,s_2,\ldots,s_k}(x)$ is a real reciprocal polynomial of degree $2s_k - 2$. Now we may assume, without loss of generality, that $f_{s_1,s_2,\ldots,s_k}(x)$ has $2v$ real roots and $4u$ complex roots. So $2v + 4u = 2s_k - 2$ and

$$\begin{aligned}\prod_{j=1}^{n-1} f_{s_1,s_2,\ldots,s_k}(\varepsilon^j) &= \prod_{j=1}^{n-1}\left\{\prod_{i=1}^{v}(\varepsilon^j-\alpha_i)(\varepsilon^j-\alpha_i^{-1})\prod_{i=1}^{u}(\varepsilon^j-\beta_i)(\varepsilon^j-\beta_i^{-1})(\varepsilon^j-\bar{\beta}_i)(\varepsilon^j-\bar{\beta}_i^{-1})\right\}\\ &= \prod_{i=1}^{v}\frac{(1-\alpha_i^n)(1-\alpha_i^{-n})}{(1-\alpha_i)(1-\alpha_i^{-1})}\prod_{i=1}^{u}\frac{(1-\beta_i^n)(1-\beta_i^{-n})(1-\bar{\beta}_i^{\,n})(1-\bar{\beta}_i^{\,-n})}{(1-\beta_i)(1-\beta_i^{-1})(1-\bar{\beta}_i)(1-\bar{\beta}_i^{\,-1})}\\ &= \frac{(-1)^{v+2u}}{f_{s_1,s_2,\ldots,s_k}(1)}\prod_{i=1}^{v}\frac{(1-\alpha_i^n)^2}{\alpha_i^n}\prod_{i=1}^{u}\frac{[(1-\beta_i^n)(1-\bar{\beta}_i^{\,n})]^2}{\|\beta_i\|^{2n}}.\end{aligned}$$

Since $v + 2u = s_k - 1$, v is even (or odd) if and only if s_k is odd (or even) and since $\alpha_i < 0$ for all $1 \leqslant i \leqslant v$, we have

$$a_n^2 = \frac{(-1)^{n(s_k-1+v)}}{f_{s_1,s_2,\ldots,s_k}(1)}\left\{\frac{\prod_{i=1}^{v}(1-\alpha_i^n)\prod_{i=1}^{u}(1-\beta_i^n)(1-\bar{\beta}_i^{\,n})}{\left(\sqrt{|\alpha_1\cdots\alpha_v|}\|\beta_1\cdots\beta_u\|\right)^n}\right\}^2, \tag{8}$$

where $|a|$ stands for the absolute value of a real number a and $\|c\|$ for the modulus of a complex number c. Let

$$\begin{aligned}&\frac{\prod_{i=1}^{v}(x-\alpha_i^n)\prod_{i=1}^{u}(x-\beta_i^n)(x-\bar{\beta}_i^{\,n})}{\left(\sqrt{|\alpha_1\cdots\alpha_v|}\|\beta_1\cdots\beta_u\|\right)^n}\\ &\quad = w_0x^{s_k-1} + w_1x^{s_k-2} + \cdots + w_{s_k-2}x + w_{s_k-1}.\end{aligned}$$

Suppose

$$r_{i,j,k}^n := \left[\frac{\alpha_{i_1}\cdots\alpha_{i_\gamma}\beta_{j_1}\cdots\beta_{j_\delta}\bar{\beta}_{k_1}\cdots\bar{\beta}_{k_\rho}}{\sqrt{|\alpha_1\cdots\alpha_v|}\|\beta_1\cdots\beta_u\|}\right]^n, \quad i_\gamma + j_\delta + k_\rho = l, \tag{9}$$

is a term of w_l for $0 \leqslant l \leqslant s_k - 1$. Then from *Vieta formula*, there is another term

$$r_{q,p,d}^n := \left[\frac{\alpha_{q_1}\cdots\alpha_{q_\mu}\beta_{p_1}\cdots\beta_{p_\nu}\bar{\beta}_{d_1}\cdots\bar{\beta}_{d_\tau}}{\sqrt{|\alpha_1\cdots\alpha_v|}\|\beta_1\cdots\beta_u\|}\right]^n, \quad q_\mu + p_\nu + d_\tau = s_k - 1 - l, \tag{10}$$

of w_{s_k-1-l} such that

$$\begin{aligned}&(\alpha_{i_1},\ldots,\alpha_{i_\gamma},\beta_{j_1},\ldots,\beta_{j_\delta},\bar{\beta}_{k_1},\ldots,\bar{\beta}_{k_\rho})\\ &\quad\cap(\alpha_{q_1},\ldots,\alpha_{q_\mu},\beta_{p_1},\ldots,\beta_{p_\nu},\bar{\beta}_{d_1},\ldots,\bar{\beta}_{d_\tau}) = \phi,\end{aligned}$$

$$\begin{aligned}&(\alpha_{i_1},\ldots,\alpha_{i_\gamma},\beta_{j_1},\ldots,\beta_{j_\delta},\bar{\beta}_{k_1},\ldots,\bar{\beta}_{k_\rho})\cup(\alpha_{q_1},\ldots,\alpha_{q_\mu},\beta_{p_1},\ldots,\beta_{p_\nu},\bar{\beta}_{d_1},\ldots,\bar{\beta}_{d_\tau})\\ &\quad = (\alpha_1,\ldots,\alpha_v,\beta_1,\ldots,\beta_u,\bar{\beta}_1,\ldots,\bar{\beta}_u),\end{aligned}$$

where ϕ represents the empty set, so

$$r_{i,j,k}^n r_{q,p,d}^n = \left[\frac{\alpha_1\cdots\alpha_v\|\beta_1\cdots\beta_u\|^2}{|\alpha_1\cdots\alpha_v|\|\beta_1\cdots\beta_u\|^2}\right]^n = (-1)^{vn}.$$

Thus considering the signs of w_l and w_{s_k-1-l}, we have

$$\begin{aligned}(-1)^l r_{i,j,k}^n + (-1)^{s_k-1-l} r_{q,p,d}^n &= (-1)^l r_{i,j,k}^n + (-1)^{s_k-1-l}(-1)^{vn} r_{i,j,k}^{-n}\\ &= (-1)^l\{r_{i,j,k}^n + (-1)^{s_k-1}[(-1)^v r_{i,j,k}]^{-n}\}.\end{aligned}$$

Now noting that v is even iff s_k is odd, this proves the lemma. □

Remark 1. Theorem 3.1 of [7] shows the reciprocal (or anti-reciprocal) properties of linear recurrence relation of a_n, where the proof is by making use of the roots of polynomial (2). Here we show the result from the characteristic polynomial of b_n. The reason 'we reprove it here' is that our idea will play an essential role in getting the new results.

Lemma 4. *For any integer* $1 \leqslant s_1 < s_2 < \cdots < s_k \leqslant \lfloor n/2 \rfloor$, *let*

$$b_n^2 = (-1)^{(s_k-1)n}|I + M_{s_1,s_2,\ldots,s_k}^n|,$$

then

$$b_n = \sum_{i=1}^{2^{s_k-2}} \begin{cases} r_i^n + r_i^{-n}, & s_k \text{ is odd},\\ r_i^n + (-r_i)^{-n}, & s_k \text{ is even}.\end{cases}$$

Proof. As before we assume that α_i, $1 \leqslant i \leqslant v$ and β_j for $1 \leqslant j \leqslant u$ are real and complex roots of the reciprocal polynomial $f_{s_1,s_2,\ldots,s_k}(x)$, respectively. Then

$$\begin{aligned}(-1)^{(s_k-1)n}|I + M_{s_1,s_2,\ldots,s_k}^n| &= (-1)^{(s_k-1)n}\prod_{i=1}^{v}(1+\alpha_i^n)(1+\alpha_i^{-n})\prod_{i=1}^{u}(1+\beta_i^n)(1+\beta_i^{-n})(1+\bar{\beta}_i^n)(1+\bar{\beta}_i^{-n})\\ &= (-1)^{(s_k-1)n}\prod_{i=1}^{v}\frac{(1+\alpha_i^n)^2}{\alpha_i^n}\prod_{i=1}^{u}\frac{[(1+\beta_i^n)(1+\bar{\beta}_i^{\,n})]^2}{\|\beta_i\|^{2n}}.\end{aligned}$$

Since $\alpha_i < 0$, $1 \leqslant i \leqslant v$ and $(-1)^v = (-1)^{s_k-1}$, we have

$$\begin{aligned}(-1)^{(s_k-1)n}|I + M_{s_1,s_2,\ldots,s_k}^n| &= (-1)^{(s_k-1+v)n}\left\{\frac{\prod_{i=1}^{v}(1+\alpha_i^n)\prod_{i=1}^{u}(1+\beta_i^n)(1+\bar{\beta}_i^{\,n})}{\left(\sqrt{|\alpha_1\cdots\alpha_v|}\|\beta_1\cdots\beta_u\|\right)^n}\right\}^2\\ &= \left\{\frac{\prod_{i=1}^{v}(1+\alpha_i^n)\prod_{i=1}^{u}(1+\beta_i^n)(1+\bar{\beta}_i^{\,n})}{\left(\sqrt{|\alpha_1\cdots\alpha_v|}\|\beta_1\cdots\beta_u\|\right)^n}\right\}^2.\end{aligned}$$

The remaining part of the proof is similar to that of Lemma 3. □

Lemma 5. a_n *and* b_n, *in Lemmas* 3 *and* 4, *share the same reciprocal* (*or anti-reciprocal*) *characteristic polynomial*:

$$p(x) = \begin{cases} \displaystyle\sum_{i=0}^{2^{s_k-2}-1} c_i(x^{2^{s_k-1}-i} + x^i) + c_{2^{s_k-2}}x^{2^{s_k-2}}, & s_k \text{ is odd},\\ \displaystyle\sum_{i=0}^{2^{s_k-2}-1} c_i(x^{2^{s_k-1}-i} + (-1)^i x^i) + c_{2^{s_k-2}}x^{2^{s_k-2}}, & s_k \text{ is even}.\end{cases}$$

Proof. From Lemmas 3 and 4, we see that the two sequences, a_n and b_n, have the same characteristic polynomial as below,

$$p(x)=\begin{cases}\prod_{i=1}^{2^{s_k-2}}(x-r_i)(x-r_i^{-1}), & s_k \text{ is odd},\\ \prod_{i=1}^{2^{s_k-2}}(x-r_i)(x-(-r_i^{-1})), & s_k \text{ is even}.\end{cases}$$

Now if s_k is odd, then

$$\begin{aligned}p(x)&=\prod_{i=1}^{2^{s_k-2}}(x-r_i)(x-r_i^{-1})\\&=x^{2^{s_k-1}}p(x^{-1}).\end{aligned}$$

This indicates that $p(x)$ is a reciprocal polynomial [13]. Similarly, if s_k is even, then

$$\begin{aligned}p(x)&=\prod_{i=1}^{2^{s_k-2}}(x-r_i)(x-(-r_i^{-1}))\\&=x^{2^{s_k-1}}p(-x^{-1}),\end{aligned}$$

i.e., $p(x)$ is anti-reciprocal. □

From Table 2 of [17], we see that the recurrence formulas for a_n have coefficients like $\sqrt{c}$, where c is a positive integer. Following Corollary 1 implies that such numbers can also happen in the general formulas.

Corollary 1. *Let*

$$T(C_n^{s_1,s_2,\ldots,s_k})=na_n^2.$$

Then $\sqrt{|f_{s_1,s_2,\ldots,s_k}(-1)|}$ *is a factor of* a_{2n}.

Proof. From the proof of Lemma 3

$$\begin{aligned}a_{2n}^2&=(-1)^{(s_k-1)(2n-1)}\prod_{j=1}^{2n-1}f_{s_1,s_2,\ldots,s_k}(\varepsilon^j)\\&=(-1)^{(s_k-1)(2n-1)}\prod_{i=1}^{n-1}f_{s_1,s_2,\ldots,s_k}(\varepsilon^i)f_{s_1,s_2,\ldots,s_k}(\varepsilon^{2n-i})f_{s_1,s_2,\ldots,s_k}(\varepsilon^n)\\&=(-1)^{(s_k-1)(2n-1)}\prod_{i=1}^{n-1}f_{s_1,s_2,\ldots,s_k}(\varepsilon^i)\overline{f_{s_1,s_2,\ldots,s_k}(\varepsilon^i)}f_{s_1,s_2,\ldots,s_k}(-1)\\&=|f(-1)|\prod_{i=1}^{n-1}\|f_{s_1,s_2,\ldots,s_k}(\varepsilon^i)\|^2.\qquad\square\end{aligned}$$

3. Simplification of the formulae

In this section, we will simplify the calculations to find the recurrence relation of a_n where we avoid solving a system of linear equations. To illustrate the technique introduced we will give two examples.

Theorem 6. *Given integers* $1\leqslant s_1<s_2<\cdots<s_k\leqslant\lfloor n/2\rfloor$, *in the formula*

$$T(C_n^{s_1,s_2,\ldots,s_k})=na_n^2$$

derived in [17], a_n *satisfies a recurrence relation of the form*

$$\sum_{i=0}^{2^{s_k-2}-1} c_i(a_{n-i}+a_{n-2^{s_k-1}+i})+c_{2^{s_k-2}}a_{n-2^{s_k-2}}=0, \quad s_k \text{ is odd},$$

$$\sum_{i=0}^{2^{s_k-2}-1} c_i(a_{n-i}+(-1)^i a_{n-2^{s_k-1}+i})+c_{2^{s_k-2}}a_{n-2^{s_k-2}}=0, \quad s_k \text{ is even},$$

where

$$c_i = -\frac{1}{i}(b_i+c_1b_{i-1}+c_2b_{i-2}+\cdots+c_{i-1}b_1), \quad c_0=1,$$
$$b_i^2=(-1)^{(s_k-1)i}|I+M^i_{s_1,s_2,\ldots,s_k}|, \quad i=1,2,\ldots,2^{s_k-2}.$$

Proof. Applying *Newton*'s identities for b_n

$$b_i+c_1b_{i-1}+c_2b_{i-2}+\cdots+c_{i-1}b_1+ic_i=0, \quad i=1,2,\ldots,2^{s_{k-1}}.$$

Thus

$$c_i=-\frac{1}{i}(b_i+c_1b_{i-1}+c_2b_{i-2}+\cdots+c_{i-1}b_1), \quad i=1,2,\ldots,2^{s_k-1}.$$

By Lemma 5, the theorem follows. □

Remark 2. For any integer $1\leqslant s_1<s_2<\cdots<s_k\leqslant\lfloor n/2\rfloor$, to find the recurrence relations, Both Theorem 8 of [17] and Theorem 3.1 of [7] need to calculate 2^{s_k} (2^{s_k-1}) values of a_n and then solve a system of 2^{s_k-1} (2^{s_k-2}) linear equations with unsymmetric Toeplitz matrix. Because of the exponential size, it is hard to solve such an unsymmetric Toeplitz system for a large k and the stability of the process cannot be assured unless its leading principle submatrices are sufficiently well conditioned [9]. Theorem 6 claims that it is not necessary to solve such a system of linear equations.

Following Examples 1 and 2 are two of the results obtained in [17]. We examine them here by using Theorem 6.

Example 1 (*Case 1*). Let $s_1=1, s_2=2, s_3=3, s_4=5$. Then

$$T(C_n^{1,2,3,5})=na_n^2,$$

where a_n satisfies the recurrence relation:

$$(a_n+a_{n-16})-\sqrt{3}(a_{n-1}+a_{n-15})-(a_{n-2}+a_{n-14})-\sqrt{3}(a_{n-3}+a_{n-13})$$
$$+(a_{n-4}+a_{n-12})-9\sqrt{3}(a_{n-5}+a_{n-11})+17(a_{n-6}+a_{n-10})+\sqrt{3}(a_{n-7}+a_{n-9})$$
$$+a_{n-8}=0$$

with initial conditions

$$a_1=1,\ a_2=\sqrt{3},\ a_3=3,\ a_4=5\sqrt{3},\ a_5=11,\ a_6=27\sqrt{3},\ a_7=113,$$
$$a_8=155\sqrt{3},\ a_9=729,\ a_{10}=979\sqrt{3},\ a_{11}=4531,\ a_{12}=6615\sqrt{3},$$
$$a_{13}=28717,\ a_{14}=42601\sqrt{3},\ a_{15}=185163,\ a_{16}=272645\sqrt{3}.$$

Proof. Since $s_4=5$ is odd, from Theorem 6, there exist c_i, $1\leqslant i\leqslant 8$ such that

$$\sum_{i=0}^{7} c_i(a_{n-i}+a_{n+i-16})+c_8a_{n-8}=0,$$

where

$$c_i = -\frac{1}{i}(b_i + c_1 b_{i-1} + c_2 b_{i-2} + \cdots + c_{i-1} b_1), \quad c_0 = 1,$$

$$b_i^2 = |I + M_{1,2,3,5}^i|, \quad i = 1, 2, \ldots, 8,$$

and

$$b_1 = \sqrt{3},\ b_2 = 5,\ b_3 = 9\sqrt{3},\ b_4 = 31,\ b_5 = 89\sqrt{3},\ b_6 = 245,\ b_7 = 377,\ b_8 = 1759,$$

thus

$$c_1 = -b_1 = -\sqrt{3},$$

$$c_2 = -\tfrac{1}{2}(b_2 + c_1 b_1) = -1,$$

$$c_3 = -\tfrac{1}{3}(b_3 + c_1 b_2 + c_2 b_1) = -\sqrt{3},$$

$$c_4 = -\tfrac{1}{4}(b_4 + c_1 b_3 + c_2 b_2 + c_3 b_1) = 1,$$

$$c_5 = -\tfrac{1}{5}(b_5 + c_1 b_4 + c_2 b_3 + c_3 b_2 + c_4 b_1) = -9\sqrt{3},$$

$$c_6 = -\tfrac{1}{6}(b_6 + c_1 b_5 + c_2 b_4 + c_3 b_3 + c_4 b_2 + c_5 b_1) = 17,$$

$$c_7 = -\tfrac{1}{7}(b_7 + c_1 b_6 + c_2 b_5 + c_3 b_4 + c_4 b_3 + c_5 b_2 + c_6 b_1) = \sqrt{3},$$

$$c_8 = -\tfrac{1}{8}(b_8 + c_1 b_7 + c_2 b_6 + c_3 b_5 + c_4 b_4 + c_5 b_3 + c_6 b_2 + c_7 b_1) = 1.$$

Then, we get the recurrence relation of a_n. The initial values a_i for $i \leqslant 16$ can easily be calculated by Lemma 2. Note that the results are same as the ones in [17]. □

Example 2 (*Case 2*). Let $s_1 = 2, s_2 = 3, s_3 = 4$. Then

$$T(C_n^{2,3,4}) = n a_n^2,$$

where a_n satisfies the recurrence relation:

$$(a_n + a_{n-8}) - (a_{n-1} - a_{n-7}) - (a_{n-2} + a_{n-6}) - 2(a_{n-3} - a_{n-5}) - 4a_{n-4} = 0,$$

with initial conditions

$$a_1 = 1,\ a_2 = 1,\ a_3 = 2,\ a_4 = 3,\ a_5 = 11,\ a_6 = 20,\ a_7 = 43,\ a_8 = 93.$$

Proof. Since $s_3 = 4$ is odd, from Theorem 6, there exist $c_i,\ 1 \leqslant i \leqslant 8$ such that

$$\sum_{i=0}^{3} c_i (a_{n-i} + a_{n+i-8}) + c_4 a_{n-4} = 0,$$

where

$$c_i = -\frac{1}{i}(b_i + c_1 b_{i-1} + c_2 b_{i-2} + \cdots + c_{i-1} b_1), \quad c_0 = 1,$$

$$b_i^2 = (-1)^i |I + M_{2,3,4}^i|, \quad i = 1, 2, \ldots, 4,$$

and

$$b_1 = 1,\ b_2 = 3,\ b_3 = 10,\ b_4 = 31,$$

thus

$$c_1 = -b_1 = -1,$$

$$c_2 = -\tfrac{1}{2}(b_2 + c_1 b_1) = -1,$$

$$c_3 = -\tfrac{1}{3}(b_3 + c_1 b_2 + c_2 b_1) = -2,$$

$$c_4 = -\tfrac{1}{4}(b_4 + c_1 b_3 + c_2 b_2 + c_3 b_1) = -4.$$

So, we have the recurrence relation of a_n. The initial values a_i for $i \leqslant 8$ can easily be calculated by Lemma 2. □

4. Asymptotic properties

This section will consider the asymptotic properties of $T(C_n^{s_1,s_2,\ldots,s_k}) = na_n^2$. Without loss of generality, in the proof of Lemma 3 we may assume that α_i, $|\alpha_i| \geqslant 1$, and β_j, $\|\beta_j\| \geqslant 1$, $1 \leqslant i \leqslant v$, $1 \leqslant j \leqslant u$, are the real, and the complex roots, respectively, of $f_{s_1,s_2,\ldots,s_k}(x)$. From (9) or (10)

$$\max_{i,j,k} \|r_{i,j,k}\| = \left\| \frac{\alpha_1 \cdots \alpha_v \beta_1 \cdots \beta_u \bar{\beta}_1 \cdots \bar{\beta}_u}{\sqrt{|\alpha_1 \cdots \alpha_v|}\|\beta_1 \cdots \beta_u\|} \right\| = \sqrt{|\alpha_1 \cdots \alpha_v|}\|\beta_1 \cdots \beta_u\|.$$

We note from Section 2 that the roots of $f_{s_1,s_2,\ldots,s_k}(x)$ are determined by $k, s_1, s_2, \ldots, s_k$. Therefore, combining all the above observations we have the following theorem.

Theorem 7. *For any integer* $1 \leqslant s_1 < s_2 < \cdots < s_k \leqslant \lfloor n/2 \rfloor$, *let*

$$T(C_n^{s_1,s_2,\ldots,s_k}) = na_n^2,$$

and let $\phi(k, s_1, s_2, \ldots, s_k) = (\sqrt{|\alpha_1 \cdots \alpha_v|}\|\beta_1 \cdots \beta_u\|)^n$. *Then*

$$a_n \sim \frac{1}{\sqrt{f_{s_1,s_2,\ldots,s_k}(1)}} \left(\sqrt{|\alpha_1 \cdots \alpha_v|}\|\beta_1 \cdots \beta_u\|\right)^n = c(k, s_1, s_2, \ldots, s_k)\phi(k, s_1, s_2, \ldots, s_k)^n.$$

Therefore,

$$T(C_{n+1}^{s_1,s_2,\ldots,s_k})/T(C_n^{s_1,s_2,\ldots,s_k}) \sim \phi(k, s_1, s_2, \ldots, s_k)^2.$$

Theorem 7 'answers' the problem posed in the conclusion of [17]. That is, the asymptotics depends continuously on all these parameters $k, s_1, s_2, \ldots, s_k$. This phenomena is much different than that of directed graphs. As an example, we consider the asymptotics for the case $s_1 = 1$, $s_2 = 2$ (i.e., the conjecture stated in the Introduction). The cases with $s_k \leqslant 7$ are in Table 1. Now

$$T(C_n^{1,2}) = na_n^2$$

and from (2), we have $f_{1,2}(x) = x^2 + 3x + 1$ and its two roots are $(-3 \pm \sqrt{5})/2$. Thus,

$$a_n \sim \frac{1}{\sqrt{5}} \left(\sqrt{\left| \frac{-3-\sqrt{5}}{2} \right|} \right)^n.$$

Therefore

$$T(C_{n+1}^{1,2})/T(C_n^{1,2}) \sim \frac{3+\sqrt{5}}{2}$$

Table 1

ϕ	s_i	ϕ	s_i	ϕ	s_i	ϕ	s_i
1.61803399	12	2.21388595	236	2.58003054	2346	2.88439537	12 346
1.70001578	13	2.21535149	347	2.58132650	1237	2.89752345	12 356
1.72208381	23	2.21674111	257	2.58132999	1346	2.90081938	12 347
1.73681478	14	2.21913771	167	2.58495976	1256	2.90480960	12 456
1.75487767	34	2.21936609	127	2.58749638	1347	2.91144102	12 357
1.75560169	15	2.22497867	345	2.59064813	1456	2.91194604	13 456
1.75957571	25	2.22546063	346	2.59097390	1356	2.91668245	23 456
1.76439390	35	2.22748546	146	2.59265011	1247	2.91986035	12 457
1.76627105	16	2.23046662	247	2.59329045	2356	2.92255935	12 367
1.76904558	45	2.23108749	356	2.59565722	2357	2.92330933	23 457
1.77285384	17	2.23128154	137	2.59652355	2347	2.92371034	13 457
1.77396566	27	2.23455013	237	2.59704745	1257	2.92779189	12 467
1.77552960	37	2.23725870	147	2.59809635	2456	2.92809814	13 467
1.77638472	56	2.23972937	256	2.60289571	1267	2.93001800	12 567
1.77728486	47	2.24041466	367	2.60582682	2457	2.93297292	23 467
1.77903434	57	2.24307230	157	2.60599726	1567	2.93837051	13 567
1.78065992	67	2.24399469	456	2.60781333	1367	2.93920003	14 567
2.10225602	123	2.24662299	357	2.61044713	1457	2.94046747	23 567
2.14739605	124	2.25259460	267	2.61067266	3456	2.94367717	24 567
2.16578607	134	2.25433519	457	2.61070702	1357	2.95034039	34 567
2.18193485	234	2.25515606	467	2.61226280	2467	3.18706665	123 456
2.18313670	125	2.25859047	567	2.61293729	3467	3.19952824	123 457
2.18979819	235	2.50960078	1234	2.61481462	3457	3.20923190	123 467
2.19475018	145	2.53709020	1235	2.61607115	1467	3.21678951	123 567
2.20050981	135	2.55525899	1245	2.61788388	2367	3.22104828	124 567
2.20421128	126	2.56083807	1236	2.62241458	2567	3.22552282	134 567
2.21009529	156	2.56361164	1345	2.62926105	3567	3.22856875	234 567
2.21039547	136	2.57002076	1246	2.63477666	4567	3.48037415	1 234 567
2.21148478	245	2.57203214	2345	2.86640386	12 345		

The values of c and ϕ for $\gcd(s_1, s_2, \ldots, s_k) = d \neq 1$ case are not reported since, as described in [17], $T(C_n^{s_1,s_2,\ldots,s_k}) = 0$ for $(n, d) \neq 1$ and $T(C_n^{s_1,s_2,\ldots,s_k}) = T(C_n^{s_1/d,s_2/d,\ldots,s_k/d})$ for $(n, d) = 1$.

(unlike the directed case, the asymptotics is not equal to 4, the degree of the vertices of the graph). We should point out that in considering the asymptotics, all we need to do is to find the products of the roots of polynomial (2) with modulus greater than 1. Table 1 is the numerical results of $c(k, s_1, s_2, \ldots, s_k)$ and $\phi(k, s_1, s_2, \ldots, s_k)$ for all possible cases of $s_k \leqslant 7$.

5. Conclusion

In this paper, we simplified the work to find the formulas for the number of spanning trees of a circulant graph. We showed that it is not necessary to solve a system of linear equations as described in [1,7,14,16,17] in determining the recurrence relations of spanning trees in circulant graphs $C_n^{s_1,s_2,\ldots,s_k}$. The asymptotics of the numbers for these graphs are proven to be dependent continuously upon the parameters $k, s_1, s_2, \ldots, s_k$. An interesting question would be to find out/estimate their exact values.

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