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Note

On the distribution of eigenvalues of graphs

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Abstract

Let G be a simple undirected graph with $n \geqslant 2$ vertices and let $\alpha_0(G) \geqslant \ldots, \alpha_{n-1}(G)$ be the eigenvalues of the adjacency matrix of G. It is shown by Cao and Yuen (1995) that if $\alpha_1(G) = -1$ then G is a complete graph, and therefore $\alpha_0(G) = n-1$ and $\alpha_i(G) = -1$ for $1 \leqslant i \leqslant n-1$. We obtain similar results for graphs whose complement is bipartite. We show in particular, that if the complement of G is bipartite and there exists an integer k such that $1 \leqslant k < (n-1)/2$ and $\alpha_k(G) = -1$ then $\alpha_i(G) = -1$ for $k \leqslant i \leqslant n-k+1$. We also compare and discuss the relation between some properties of the Laplacian and the adjacency spectra of graphs. © 1999 Elsevier Science B.V. All rights reserved

1. Introduction

We consider undirected graphs having no loops or parallel edges. All notions on graphs that are not defined here can be found in [1].

Let $V(G) = \{v_1, \ldots, v_n\}$ be the set of vertices of a graph G. Let $d_G(v)$ denote the degree of a vertex v in G. We assume that $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$ (and so $n > 0$).

Let $A(G) = \{a_{ij}\}$ where $a_{ij} = 1$ if $v_i v_j \in E(G)$ and $a_{ij} = 0$ if $v_i v_j \notin E(G)$. Let $D(G) = \{d_{ij}\}$ where $d_{ij} = d_G(v_i)$ if $i = j$ and $d_{ij} = 0$ if $i \neq j$, $i, j \in \{i, \ldots, n\}$, i.e. $D(G)$ is the diagonal matrix with the degrees of the corresponding vertices of G on the main diagonal. Let $L(G) = D(G) - A(G)$.

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The matrices $A(G)$ and $L(G)$ are called the *adjacency matrix* and the *Laplacian matrix* of G, respectively.

Many papers are devoted to study the characteristic polynomials and spectra of the adjacency and Laplacian matrices of a graph and their possible relations with various properties of the graph (see, for example, [2–18]).

In Section 2 we compare and discuss the relation between some properties of the Laplacian and the adjacency spectra of graphs.

In Section 3 we use some facts described in Section 2 to prove some new results concerning the distribution of the adjacency eigenvalues of graphs.

2. Comparison of properties of the Laplacian and adjacency spectra of graphs

Since both $A(G)$ and $L(G)$ are symmetric matrices, clearly their eigenvalues are real numbers.

Let $\alpha_0(G) \geqslant \cdots \geqslant \alpha_{n-1}(G)$ and $\lambda_0(G) \leqslant \cdots \leqslant \lambda_{n-1}(G)$ be the spectrum of $A(G)$ and $L(G)$, respectively. If G is a r-regular graph then clearly $L(G) = rI - A(G)$, and so $\lambda_i(G) = r - \alpha_i(G)$. Therefore many results on the Laplacian spectrum of graphs can be translated into the adjacency spectrum language and vice versa. For non-regular graphs the situation turns out to be quite different.

Theorem 2.1 (Coulson and Rushbrooke [4] and Sachs [18]). *A graph G is bipartite if and only if*

$$\alpha_i(G) + \alpha_{n-i-1}(G) = 0 \quad \text{for } i \in \{0, \ldots, n-1\}.$$

This theorem is an important result establishing a connection between the structure and adjacency spectra of graphs. It is known in chemistry as the 'paring theorem' [5].

From Theorem 2.1 we have for regular graphs:

Theorem 2.2. *An r-regular graph G is bipartite if and only if*

$$\lambda_i(G) + \lambda_{n-i-1}(G) = 2r \quad \text{for } i \in \{0, \ldots, n-1\}.$$

Let G be a graph with n vertices, $\bar{G}$ denote the graph complement to G.

Theorem 2.3 (Kelmans [8,9]). *Let $i \in \{1, \ldots, n-1\}$. Then*

(L) $\lambda_i(G) + \lambda_{n-i}(\bar{G}) = n$,

and therefore

(A) *if G is a regular graph then* $\alpha_i(G) + \alpha_{n-i}(\bar{G}) = -1$.

Theorem 2.3 (and its generalization on weighted graphs [11]) opens various opportunities. It allows to find a simple algorithm that provides formulas for the Laplacian polynomial and spectrum, and the number of spanning trees of so called decomposable graphs [8,9,11]. One of many applications of this algorithm is a description of the Laplacian spectrum and the number of spanning trees of a threshold graph in terms of the vertex degrees of the graph [6]. Theorem 2.3 also allows to establish the so called inclusion–exclusion properties of the Laplacian polynomial of a graph [10,13]. Comparison of graphs by their number of spanning trees and finding graphs with an extremal (maximum or minimum) number of spanning trees among graphs of a certain type is an important problem in the extremal graph theory and network reliability. One of the approaches to this problem is based on Theorem 2.3 [13–16].

It is known and it is easy to see that

Theorem 2.4 (Kelmans [8,9], see also Cvetkovic et al. [5]). *Let G be a graph. Then*

(L1) *$L(G)$ is positive semi-definite and therefore $\lambda_0(G) \geqslant 0$,*

(L2) *$\lambda_0(G) = 0$ and the multiplicity of the Laplacian eigenvalue 0 is equal to the number of components of G,*

and therefore

(A) *if G is r-regular, then $\alpha_0(G) = r$ and the multiplicity of the adjacency eigenvalue r is equal to the number of components of G.*

From Theorems 2.3 and 2.4 it follows that

Theorem 2.5 (Kelmans [8,9], see also Cvetkovic et al. [5]). *Let $i \in \{1, \ldots, n-1\}$. Then*

(L) $0 \leqslant \lambda_i(G) \leqslant n$,

and therefore

(A) *if G is a regular graph of degree r then $r - n \leqslant \alpha_i(G) \leqslant r$.*

Let $\Delta(G)$ and $\delta(G)$ denote the maximum and the minimum vertex degree in G, respectively. Let $\lambda_{\max}(G) = \lambda_{n-1}(G)$ and $\lambda_{\min}(G) = \lambda_1(G)$.

Theorem 2.6 (Kelmans and Chelnokov [13]). *Let G be a graph with $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$. Then*

(L1) $\Delta(G) + 1 \leqslant \lambda_{\max}(G) \leqslant \min\{n, \max\{d(u,G)) + d(v,G)\colon\ u, v \in V(G), u \neq v\}\} \leqslant \min\{n, 2\Delta(G)\}$, *and $\lambda_k(G) \geqslant 1$ for $k \in \{n - \Delta(G), \ldots, n-1\}$,*

(L2) *if G is not a complete graph then $\lambda_{\min}(G) \leqslant \delta(G)$,*

and therefore for r-regular graph G we have:

(A1) $\max\{r - n, -r\} \leqslant \alpha_{n-1}(G) \leqslant -1$,

(A2) *if G is not a complete graph, then $\alpha_1(G) \geqslant 0$,*

(A3) *if G is a complete graph with at least 2 vertices, then $\alpha_1(G) = -1$ (and therefore there is no regular graph with $\alpha_1(G) \in (-1, 0)$), and*

(A4) $\alpha_k(G) \leqslant r - 1$ *for $k \in \{n - \Delta(G), \ldots, n-1\}$.*

It is easy to see that

Theorem 2.7 (Kelmans [12]). *Let G be a connected graph with at least 2 vertices. Then* $\lambda_{\max}(G) = \Delta(G) + 1$ *if and only if* $|V(G)| = \Delta(G) + 1$.

From Theorem 2.7 we have

Theorem 2.8 (Kelmans [12]). *Let G be a graph with at least 2 vertices. Then* $\lambda_{\max}(G) = \Delta(G) + 1$ *if and only if*

(c) $|V(D)| = \Delta(G)+1$ *for every component D of G containing a vertex of maximum degree and* $\lambda_{\max}(C) \leqslant \Delta(G) + 1$ *for every component C of G that contains no vertex of maximum degree.*

From Theorems 2.3 and 2.8. We have

Theorem 2.9 (Kelmans [12]). *Let G be a non-complete graph. Then* $\lambda_{\min}(G) = \delta(G)$ *if and only if* $\bar{G}$ *satisfies condition* (c) *in Theorem* 2.8.

From Theorem 2.9 we have for the adjacency spectrum:

Theorem 2.10 (Kelmans [12]). *Let G be a r-regular graph with* $n \geqslant 2$ *vertices. Then* $\lambda_1(G) = r$ *(or equivalently,* $\alpha_1(G) = 0$*) if and only if G is a complete k-partite graph with* $k < n$.

It turns out that Theorems 2.6(A2) and 2.10 hold not only for regular graphs.

Theorem 2.11 (Cao and Yuen [12]). *Let G be a graph with* $n \geqslant 2$ *vertices. Then*

(A1) *if G is not a complete graph, then* $\alpha_1(G) \geqslant 0$,

(A2) *if G is a complete graph, then* $\alpha_1(G) = -1$ *(and therefore there is no regular graph with* $\alpha(G) \in (-1, 0)$*), and*

(A3) $\alpha_1(G) = 0$ *if and only if G is a complete k-partite graph with* $k < n$.

From Theorems 2.1 and 2.3 it follows that

Theorem 2.12. *Let G be a r-regular graph and* $\bar{G}$ *is bipartite. Then* $\alpha_{n-i-1}(\bar{G}) = \alpha_{n-i}(G) + 1$, *and in particular,* $\alpha_1(G) = n - r - 2$.

Theorem 2.13 (Cao and Yuen [3]). *Let* $i \in \{2, \ldots, n-1\}$. *Then*

$$\alpha_i(G) + \alpha_{n-i}(\bar{G}) \leqslant -1 \tag{2.1}$$

and

$$-1 \leqslant \alpha_i(G) + \alpha_{n-i-1}(\bar{G}). \tag{2.2}$$

If G is regular then the above statement follows immediately from Theorem 2.3(A). Therefore Theorem 2.13 can be interpreted as an analogue of Theorem 2.3(L) and an extension of Theorem 2.3(A) for the adjacency spectrum of graphs.

Theorem 2.14 (Cao and Yuen [3]). *Let G be a graph with n vertices. Then*

(A1) *if $\alpha_2(G) < -1$ then G is isomorphic to a path with 3 vertices,*

(A2) *if G has at least 4 vertices then $\alpha_2(G) \geqslant -1$,*

(A3) *if $\alpha_2(G) = -1$ then $\bar{G}$ is a complete bipartite graph plus possible isolated vertices,*

(A4) *if $\alpha_2(G) < 0$ then $\bar{G}$ is a bipartite graph, and*

(A5) *there is no graph G with $\alpha_2(G) \in (-1, -(\sqrt{5} - 1/2)$.*

From Theorems 2.6 and 2.14 we have for the Laplacian spectrum:

Theorem 2.15. *Let G be an r-regular graph with n vertices. Then*

(L1) $r + 1 \leqslant \lambda_{n-1}(G) \leqslant \min\{n, 2r\}$,

(L2) $r \leqslant \lambda_{n-2}(G)$,

(L3) $\lambda_2(G) \leqslant r + 1$,

(L4) $\lambda_2(G) = r + 1$ *if and only if G has exactly two components each isomorphic to a complete graph with $r + 1$ vertices,*

(L5) *if $r < \lambda_2(G)$ then $\bar{G}$ is r-regular bipartite graph,*

(L6) *there is no r-regular graph G with $\lambda_2(G) \in (r + (\sqrt{5} - 1)/2, r + 1)$,*

(L7) *if G is not a complete graph, then $\lambda_1(G) \leqslant r$, and*

(L8) *if G is a complete graph with at least 2 vertices, then $\lambda_1(G) = r + 1$ (and therefore there is no r-regular graph with $\lambda_1(G) \in (r, r + 1)$).*

3. Main results

Theorem 3.1. *Let G be a simple undirected graph with n vertices. Suppose that $\bar{G}$ is bipartite. Then*

(a1) $\alpha_k(G) \geqslant -1$ *for* $0 \leqslant k \leqslant (n-1)/2$,

(a2) $\alpha_k(G) \leqslant -1$ *for* $(n + 1)/2 \leqslant k \leqslant (n-1)$, *and*

(a3) *if $\alpha_k(G) = -1 + \delta_k$ for some $k \in \{1, \ldots, \lfloor (n-1)/2 \rfloor\}$, where $\delta_k \geqslant 0$ (see (a1)), then*

$$\alpha_{n-k+1}(G) \leqslant -1 - \delta_k \leqslant \alpha_{n-k-1}(G) \leqslant \cdots \leqslant \alpha_k(G) = -1 + \delta_k, \tag{3.1}$$

$$\alpha_{n-k}(\bar{G}) \leqslant -\delta_k \leqslant \alpha_{n-k-1}(\bar{G}) \leqslant \cdots \leqslant \alpha_k(\bar{G}) \leqslant \delta_k \leqslant \alpha_{k-1}(\bar{G}). \tag{3.2}$$

Proof. (p1) Let us first prove (a1) and (a3). Let $\alpha_k(G) = -1 + \delta_k$. By (2.1) in Theorem 2.13,

$$\alpha_k(G) + \alpha_{n-k}(\bar{G}) \leqslant -1 \tag{3.3}$$

and by (2.2) in Theorem 2.13,

$$-1 \leqslant \alpha_k(G) + \alpha_{n-k-1}(\bar{G}). \tag{3.4}$$

Since $\alpha_k(G) = -1 + \delta_k$ we have

$$\alpha_{n-k}(\bar{G}) \leqslant -\delta_k, \tag{3.5}$$

$$-\delta_k \leqslant \alpha_{n-k-1}(\bar{G}). \tag{3.6}$$

Since $\bar{G}$ is a bipartite graph, we have by Theorem 2.1(A),

$$\alpha_i(\bar{G}) = -\alpha_{n-i-1}(\bar{G}) \tag{3.7}$$

for $1 \leqslant i \leqslant n$. From (3.6) and (3.7) with $i = k$ we have

$$\alpha_k(\bar{G}) \leqslant \delta_k. \tag{3.8}$$

Suppose that $k \leqslant (n-1)/2$. Then $n-k-1 \leqslant k$ and therefore

$$\alpha_{n-k-1}(\bar{G}) \leqslant \alpha_{n-k-2}(\bar{G}) \leqslant \cdots \leqslant \alpha_k(\bar{G}), \tag{3.9}$$

$$\alpha_{n-k-1}(G) \leqslant \alpha_{n-k-2}(G) \leqslant \cdots \leqslant \alpha_k(G). \tag{3.10}$$

From (3.6), (3.8), and (3.9) we obtain

$$-\delta_k \leqslant \alpha_{n-k-1}(\bar{G}) \leqslant \alpha_{n-k-2}(\bar{G}) \leqslant \cdots \leqslant \alpha_k(\bar{G}) \leqslant \delta_k. \tag{3.11}$$

From (3.11) we have $\delta_k \geqslant 0$. Thus if $k \leqslant (n-1)/2$ then

$$\alpha_k(G) = -1 + \delta_k \geqslant -1, \tag{3.12}$$

(and so (a1) Theorem 3.1 is proved). By (3.4) in Theorem 2.13 with $i = k$ and $G := \bar{G}$,

$$-1 \leqslant \alpha_k(\bar{G}) + \alpha_{n-k-1}(G). \tag{3.13}$$

Therefore from (3.8) and (3.13), $\alpha_{n-k-1}(G) \geqslant -1-\delta_k$. Thus from (3.10)

$$-1-\delta_k \leqslant \alpha_{n-k-1}(G) \leqslant \alpha_{n-k-2}(G) \leqslant \cdots \leqslant \alpha_k(G) = -1 + \delta_k. \tag{3.14}$$

By (3.5) and (3.7) with $i = k-1$,

$$-\alpha_{k-1}(\bar{G}) = \alpha_{n-k}(\bar{G}) \leqslant -\delta_k. \tag{3.15}$$

Therefore

$$\delta_k \leqslant \alpha_{k-1}(\bar{G}). \tag{3.16}$$

By (2.1) in Theorem 2.13 with $i = k-1$ and $G := \bar{G}$,

$$\alpha_{k-1}(\bar{G}) + \alpha_{n-k+1}(G) \leqslant -1. \tag{3.17}$$

From (3.16) and (3.17) we have

$$\alpha_{n-k+1}(G) \leqslant -1-\delta_k. \tag{3.18}$$

Now (3.1) in Theorem 3.1 follows from (3.14) and (3.18), and (3.2) in Theorem 3.1 follows from (3.11), (3.15), and (3.16).

(p2) Now let us prove (a2). Suppose on the contrary that $\alpha_k(G) = -1 + \delta$ for some $\delta > 0$ and $(n+1)/2 \leqslant k \leqslant n-1$. Since $(n+1)/2 \leqslant k \leqslant n-1$, clearly $n-k \leqslant k-1$, and therefore

$$\alpha_{k-1}(\bar{G}) \leqslant \alpha_{n-k-2}(\bar{G}). \tag{3.19}$$

Now by 3.15, 3.16, and 3.19, $\delta \leqslant \alpha_{k-1}(\bar{G}) \leqslant \alpha_{n-k}(\bar{G}) \leqslant -\delta$. This contradicts the fact that $\delta > 0$. Therefore $\alpha_k(G) \leqslant -1$ if $(n+1)/2 \leqslant k \leqslant n-1$. □

Now assume that $\lambda_k(G)$ for some $k \in \{2, \ldots, [(n+1)/2]\}$ is known with a certain accuracy. Then the following modification of Theorem 3.1 (a3) describes the corresponding localization results, and can be proved by using similar arguments.

Theorem 3.2. *Let G be a simple undirected graph. Suppose that $\bar{G}$ is bipartite. Let $-1 + \delta_k'' \leqslant \alpha_k(G) \leqslant -1 + \delta_k'$ for some $k \in \{1, \ldots, [(n-1)/2]\}$, where $\delta_k'' \geqslant 0$ (and so $0 \leqslant \delta_k'' \leqslant \delta_k'$). Then*

$$\alpha_{n-k+1}(G) \leqslant -1 - \delta_k'',$$

$$-1 - \delta_k' \leqslant \alpha_{n-k-1}(G) \leqslant \cdots \leqslant \alpha_k(G) \leqslant -1 + \delta_k',$$

$$\alpha_{n-k}(\bar{G}) \leqslant -\delta_k'',$$

$$-\delta_k' \leqslant \alpha_{n-k-1}(\bar{G}) \leqslant \cdots \leqslant \alpha_k(\bar{G}) \leqslant \delta_k',$$

$$\delta_k'' \leqslant \alpha_{k-1}(\bar{G}).$$

From Theorems 2.14 (A4) and Theorem 3.1 we have

Theorem 3.3. *If $\alpha_2(G) < 0$ then the statements* (a1), (a2), *and* (a3) *of Theorem* 3.1 *and the statement of Theorem* 3.2 *hold.*

From Theorem 3.1 we have the following corollary.

Theorem 3.4. *Suppose that*

(h1) *$\bar{G}$ is bipartite (or $\alpha_2(G) < 0$) and*

(h2) *there exists an integer k such that $1 \leqslant k < (n-1)/2$ and $\alpha_k = -1$.*

Then $\alpha_i = -1$ for every $i \in \{k, k+1, \ldots, n-k+1\}$.

Proof. Follows from (3.1) in Theorem 3.1 if we put $\delta_k = 0$. □

From Theorem 3.1 we have the following corollary.

Theorem 3.5. *Let G be a graph with* $n \geqslant 6$ *vertices. Then* $\alpha_2(G) = -1$ *implies that* $\alpha_i(G) = -1$ *for* $i \in \{2, \ldots, n-3\}$.

Theorem 3.6. *Suppose that G is a disconnected graph with n vertices and* $\alpha_2(G) < 0$. *Then G has exactly two components and each of these components is a complete graph, and therefore if G has no isolated vertices then* $\alpha_i(G) = -1$ *for* $i \in \{2, \ldots, n-1\}$.

Proof. Since $\alpha_2(G) < 0$, by Theorem 2.14, $\bar{G}$ is bipartite. Since G is a disconnected graph and $\bar{G}$ is bipartite, clearly G has exactly two components and each of these components is a complete graph. It is also easy to see that if $s \geqslant 2$ then $\alpha_0(K_s) = s-1$ and $\alpha_i(K_s) = -1$ for $i \in \{1, \ldots, s-1\}$. Therefore if G has no isolated vertices then $\alpha_i(G) = -1$ for $i \in \{2, \ldots, n-1\}$. □

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