NORTH-HOLLAND

Elliptic Matrices and Their Eigenpolynomials

Xuerong Yong and Zheng Wang
Department of Mathematics
Xinjiang University
Urumqi 830046, P. R. China

Submitted by Richard A. Brualdi

ABSTRACT

The general elliptic matrices are investigated. Some properties are derived. The signs of coefficients of the eigenpolynomials are discussed. The paper generalizes some results presented by M. Fiedler. Furthermore, as an application, Smith's result that a simple connected graph is completely multipartite iff it has exactly one positive eigenvalue is reproved. © Elsevier Science Inc., 1997

1. INTRODUCTION, NOTATION, AND PRELIMINARIES

All matrices considered in this paper will be real if not otherwise stated.

A symmetric matrix is called *elliptic* if it has exactly one, simple positive eigenvalue. An elliptic matrix with all diagonal entries equal to zero is known as a special elliptic matrix [1]. We denote by $\overline{\mathscr{K}}_n$ the class of all elliptic matrices of order n, and by $\overline{\mathscr{K}}_n^+$ its subclass consisting of all nonnegative matrices in $\overline{\mathscr{K}}_n$. In addition, $\mathscr{K}_n$ stands for the class of all special elliptic matrices of order n, and $\mathscr{K}_n^+$ for the subclass of $\mathscr{K}_n$, in which elements are nonnegative matrices. M. Fiedler studied the matrices in $\mathscr{K}_n$ and provided some interesting results in [1].

655 Avenue of the Americas, New York, NY 10010

0024-3795/97/$17.00
PII S0024-3795(96)00310-2

We define a normalized general diagonal matrix as a matrix of the block form

$$\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_t \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

where the d_is, $i = 1, 2, \dots, t$, are column unit vectors with all coordinates different from zero [1].

The present paper is concerned with the matrices in $\overline{\mathscr{K}}_n$; some results may be regarded as a generalization of M. Fiedler's work in [1]. In Section 2 we shall explore the general properties of matrices in $\overline{\mathscr{K}}_n$. Their principal submatrices and the signs of coefficients of the eigenpolynomials will be considered in Section 3, in which we shall also establish the Smith's result that a simple connected graph is complete k-partite if and only if it has exactly one positive eigenvalue.

LEMMA 1.1. *Given an $n \times n$ Hermitian matrix $A = (a_{ij})$, then A is unitarily similar to a matrix each of whose main diagonal entries is equal to $(1/n)\operatorname{tr} A$.*

Proof. We shall use induction on n. For $n = 2$, in case $a_{11} \neq a_{22}$, let

$$B = (b_{ij}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ \bar{a}_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

We have

$$b_{11} = a_{11}\cos^2\theta - 2\operatorname{Re} a_{12}\cos\theta\sin\theta + a_{22}\sin^2\theta,$$

$$b_{22} = a_{11}\sin^2\theta + 2\operatorname{Re} a_{12}\cos\theta\sin\theta + a_{22}\cos^2\theta.$$

Setting $b_{11} = b_{22}$, this leads to

$$(a_{11} - a_{22})\cos 2\theta = 2\operatorname{Re} a_{12}\sin 2\theta,$$

and yields the following:

$$\theta = \tfrac{1}{2}\arctan\frac{2\operatorname{Re} a_{12}}{a_{11} - a_{22}}.$$

This implies that the assertion is true for $n = 2$. Supposing that it is valid for $n-1$, we are to consider the case for n. Let

$$Ax_i = \lambda_i x_i, \quad i = 1, 2, \dots, n, \qquad (x_i, x_j) = \begin{cases} 1, & i = j; \\ 0, & i \neq j, \end{cases}$$

and let $x = (1/\sqrt{n})(x_1 + x_2 + \cdots + x_n)$. Then we may construct a unitary matrix $V = (x, Q)$ such that

$$V^*AV = \begin{pmatrix} \dfrac{\operatorname{tr} A}{n} & * \\ * & B \end{pmatrix},$$

which gives

$$\frac{\operatorname{tr} A}{n} = \frac{\operatorname{tr} B}{n-1}.$$

According to the induction, there exists a unitary matrix V_1 such that

$$V_1^*BV_1 = \begin{pmatrix} \dfrac{\operatorname{tr} B}{n-1} & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & \dfrac{\operatorname{tr} B}{n-1} \end{pmatrix} = \begin{pmatrix} \dfrac{\operatorname{tr} A}{n} & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & \dfrac{\operatorname{tr} A}{n} \end{pmatrix}.$$

Set $U = V\begin{pmatrix} 1 & 0 \\ 0 & V_1 \end{pmatrix}$; then

$$U^*AU = \begin{pmatrix} \dfrac{\operatorname{tr} A}{n} & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & \dfrac{\operatorname{tr} A}{n} \end{pmatrix}. \qquad \blacksquare$$

COROLLARY 1.2. *Let* $A = (a_{ij}) \in \bar{\mathscr{K}}_n$ *and* $\operatorname{tr} A \geqslant 0$. *Then there exists an orthogonal matrix* Q *such that* $Q^TAQ = B = (b_{ij}) \in \bar{\mathscr{K}}_n$, *and* $b_{ii} = (\operatorname{tr} A)/n \geqslant 0$, $i = 1, 2, \dots, n$.

LEMMA 1.3. *Let*

$$p(z) = z^n - p_1 z^{n-1} - p_2 z^{n-2} - \cdots - p_{n-1} z - p_n,$$

where $p_i \geqslant 0$, $i = 1, 2, \dots, n$, *and* $\sum_{i=1}^n p_i > 0$. *Then* $p(z) = 0$ *has exactly one positive root, which is the dominant one among the n roots of* $p(z) = 0$.

Proof. The proof of the first part may be found in [2], and the second follows because the companion matrix of $p(z)$ is a nonnegative matrix of order n. ■

2. THE MATRICES IN $\overline{\mathscr{K}}_n$

We shall first present some properties of matrices in $\overline{\mathscr{K}}_n$. For a special case, this yields some related results considered in [1].

LEMMA 2.1. *The determinant of a nonsingular matrix* $A \in \overline{\mathscr{K}}_n$ *has sign* $(-1)^{n-1}$.

Proof. Noticing that the determinant is the product of the eigenvalues, the validity is trivial. ■

THEOREM 2.2. *Let m, n be integer numbers,* $m \leqslant n$. *If* $A = (a_{ij}) \in \overline{\mathscr{K}}_n$, *and* $a_{ii} \geqslant 0$, $i = 1, 2, \dots, n$, *then every* $m \times m$ *principal submatrix, say* $A_{m \times m}$, *of A either belongs to* $\overline{\mathscr{K}}_m$ *or is zero. In particular, if* $a_{ii} > 0$, $i = 1, 2, \dots, n$, *then* $A_{m \times m} \in \overline{\mathscr{K}}_m$.

Proof. In view of the interlacing theorem, every nonzero $(n-1) \times (n-1)$ principal submatrix of $A \in \overline{\mathscr{K}}_n$ has at most one positive eigenvalue. On the other hand, its trace being nonnegative, it has at least one such eigenvalue. The rest follows by induction. ■

THEOREM 2.3. *Let* $A = (a_{ij}) \in \overline{\mathscr{K}}_n$, $n \geqslant 2$, *have all diagonal entries nonnegative and all off-diagonal entries different from zero. Then there exists a diagonal matrix* $\mathrm{diag}(a_1, s_2, \dots, s_n)$, *with* $s_i = 1$ *or* -1, *such that* $SAS \in \overline{\mathscr{K}}_n^+$.

Proof. The case $n = 2$ is obvious. Let $n \geqslant 3$; for $1 < i < j$, we consider the principal minor with indices $1, i, j$.

By Lemma 2.1 and Theorem 2.2, we have

$$0 \leqslant \begin{vmatrix} a_{11} & a_{1i} & a_{1j} \\ a_{i1} & a_{ii} & a_{ij} \\ a_{j1} & a_{ji} & a_{jj} \end{vmatrix} = a_{11}a_{ii}a_{jj} + 2a_{1i}a_{1j} - a_{11}a_{ij}^2 - a_{ii}a_{1j}^2 - a_{jj}a_{1i}^2.$$

We show that $a_{1i}a_{1j}a_{ij} > 0$. In case $a_{1i}a_{1j}a_{ij} < 0$, we would have

$$0 < a_{11}a_{ii}a_{jj} - a_{11}a_{ij}^2 - a_{ii}a_{1j}^2 - a_{jj}a_{1i}^2,$$

and this implies that $a_{ii} > 0$, $i = 1, 2, \dots, n$. Therefore, by Theorem 2.2 we have the following:

$$0 < a_{11}a_{ii}a_{jj} - a_{11}a_{ij}^2 = a_{11}\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \leqslant 0.$$

This is a contradiction. It now suffices to choose $s_1 = 1$, $s_i = \operatorname{sign} a_{1i}$, $i = 2, 3, \dots, n$. ■

COROLLARY 2.4. *Let $A = (a_{ij}) \in \bar{\mathscr{K}}_n$ have all diagonal entries nonnegative and all off-diagonal entries different from zero. Then the spectral radius of A is the positive eigenvalue*

Proof. The assertion follows from Theorem 2.3 and from the Perron-Frobenius theorem [3]. ■

LEMMA 2.5. *Let $A = (a_{ij}) \in \bar{\mathscr{K}}_n$ have all diagonal entries nonnegative and the off-diagonal entry a_{pq} equal to zero. Then either the pth and qth rows are proportional of the pth row is zero.*

Proof. The assertion is trivial for $n = 2$. For the case $n = 3$, by Theorem 2.2, we have

$$0 \leqslant \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2.$$

It is then easy to see the validity of the assertion by following the proof of Theorem 2.3. Now we consider the case $n \geqslant 4$. Let r, s be any indices such that the four numbers p, q, r, s are all distinct; then the principal minor in the rows with these indices is nonpositive due to Lemma 2.1 and Theorem 2.2. Set

$$\begin{vmatrix} a_{pp} & 0 & a_{pr} & a_{ps} \\ 0 & a_{qq} & a_{qr} & a_{qs} \\ a_{rp} & a_{rq} & a_{rr} & a_{rs} \\ a_{sp} & a_{sq} & a_{sr} & a_{ss} \end{vmatrix} = -a^2 \leqslant 0;$$

then since

$$\begin{pmatrix} a_{pp} & 0 \\ 0 & a_{qq} \end{pmatrix} \in \bar{\mathscr{K}}_2$$

(or is zero), we have two situations to consider.

(1) If $a_{pp} = a_{qq} = 0$, then we have readily that

$$\begin{vmatrix} a_{pr} & a_{ps} \\ a_{qr} & a_{qs} \end{vmatrix} = 0,$$

which implies that the assertion is true for arbitrary r, s.

(2) If (1) is not the case, we can assume, without loss of generality, that $a_{pp} = 0$ and $a_{qq} > 0$; then by Lemma 2.1 and Theorem 2.2, we have $-a_{qq}a_{pr}^2 \geqslant 0$ (by considering the principal minor in the rows with indices p, q, r). Hence, $a_{pr} = 0$. Now in case $a_{rr} = 0$, this case can be changed into situation (1). So we may also suppose that $a_{rr} > 0$, and consider the following 3×3 principal submatrix:

$$\begin{pmatrix} 0 & 0 & a_{ps} \\ 0 & a_{rr} & a_{rs} \\ a_{ps} & a_{rs} & a_{ss} \end{pmatrix}.$$

As considered above, we obtain then $a_{ps} = 0$. Continuing this procedure, we find, at last, that the matrix A has the property that either the pth and qth rows are proportional or the pth row is zero. ■

COROLLARY 2.6. *Let* $A = (a_{ij}) \in \bar{\mathscr{K}}_n$, *and let* $a_{ii} \geqslant 0$, $i = 1, 2, \dots, n$. *If* A *is nonsingular, then* A *has all off-diagonal entries different from zero.*

COROLLARY 2.7. *Let* $A = (a_{ij}) \in \bar{\mathscr{K}}_n$, *and let* $a_{ii} \geqslant 0$, $i = 1, 2, \dots, n$. *If* $a_{pq} = 0$, *and* $q_{qs} = 0$ *for some* s, *then* $a_{pp} \times a_{qq} = 0$ *and* $a_{ps} = 0$.

THEOREM 2.8. *Let* $A = (a_{ij}) \in \bar{\mathscr{K}}_n$ *have rank* r, *and* $a_{ii} \geqslant 0$, $i = 1, 2, \dots, n$. *Then there exists an integer* t, $r \leqslant t \leqslant n$, *an* $n \times n$ *permutation matrix* P, *an* $n \times t$ *normalized general diagonal matrix* D, *and a matrix* $A_0 \in \bar{\mathscr{K}}_t^+$ *with rank* r *such that*

$$A = PDA_0 D^T P^T. \tag{1}$$

The matrix A_0 *is unique up to a simultaneous permutation of rows and columns.*

Conversely, if $A_0 \in \bar{\mathscr{K}}_t^+$ *has rank* r *and if* D *is a normalized* $n \times t$ *diagonal matrix and* P *an* $n \times n$ *permutation matrix, then* A *from* (1) *is an elliptic matrix in* $\bar{\mathscr{K}}_n$ *with rank* r.

Proof. This theorem can be proved by using Corollary 2.7, Lemma 2.5, and Theorem 2.3 above and Lemma A in [1]. The details are similar to the proof of Theorem 2.9 in [1]. We omit them here. ∎

3. ON THE PRINCIPAL SUBMATRICES AND THE EIGENPOLYNOMIAL OF A MATRIX IN $\bar{\mathscr{K}}_n$

THEOREM 3.1. *Let* $A = (a_{ij}) \in \bar{\mathscr{K}}_n$, *and* $a_{i_0 i_0} > 0$ *for some* i_0 $(1 \leqslant i_0 \leqslant n)$. *Then* A *has a sequence of nested principal submatrices* $\{A_i\}$ *such that* $A_i \in \bar{\mathscr{K}}_i$, $i = 1, 2, \dots, n$.

Proof. Without loss of generality, we can assume that $a_{11} > 0$. We need only show that all the leading principal submatrices of A are elliptic. For the $(n-1) \times (n-1)$ leading principal submatrix, by the interlacing theorem and noticing that $a_{11} > 0$, it suffices to see the validity. The rest follows by induction. ∎

THEOREM 3.2. *Let* $A = (a_{ij}) \in \bar{\mathscr{K}}_n$, *and* $\operatorname{tr} A \geqslant 0$, *then the eigenpolynomial of* A, *say* $p(\lambda)$, *has the form* $p(\lambda) = \lambda^n - a_1\lambda^{n-1} - a_2\lambda^{n-2} - \cdots - a_{r-1}\lambda^{n-r+1} - a_r\lambda^{n-r}$, *where* $a_1 \geqslant 0$, $a_i > 0$, $i = 2, 3, \dots, r$, *and* $r = \operatorname{rank} A$.

Proof. By Corollary 1.2 and Theorem 2.2, we need only consider the matrix B of Corollary 1.2. Letting $r = \operatorname{rank} A = \operatorname{rank} B$, it is easy to see that

$$p(\lambda) = \lambda^n - a_1\lambda^{n-1} - a_2\lambda^{n-2} - \cdots - a_{r-1}\lambda^{n-r+1} - a_r\lambda^{n-r},$$

where $a_i \geqslant 0$, $i = 1, 2, \ldots, r-1$, $a_r > 0$. The is suffices to show that $a_i > 0$, $i = 2, \ldots, r-1$. If, for some p, $2 \leqslant p \leqslant r-1$, we have that $a_p = 0$, then since all the principal minors of $p \times p$ have same sign $(-1)^{p-1}$ (if not zero), $a_p = 0$ implies that they are all zeros. Suppose that A_{p+1} is a principal submatrix of order $p+1$ of B, and A_p is a principal submatrix of A_{p+1}. Noticing that both A_p and A_{p+1} are elliptic, we know that A_{p+1} has an eigenvalue equal to zero by the interlacing theorem. Therefore, we obtain that all the principal minors of order $p+1$ are zeros, and consequently, we have that $a_p = a_{p+1} = \cdots = a_r = 0$. But $a_r > 0$. This is a contradiction. ■

THEOREM 3.3. *Let* $A = (a_{ij}) \in \bar{\mathscr{K}}_n$, $a_{ii} \geqslant 0$, $i = 1, 2, \ldots, n$, *and* $\det A \neq 0$. *Then every principal submatrix of order* r, $n - 1 \geqslant r \geqslant 2$, *of* A *is nonsingular.*

Proof. The proof follows from Theorem 2.2 and the interlacing theorem. ■

REMARK. Even if a matrix $A \in \bar{\mathscr{K}}_n$ together with every principal submatrix of order r $(2 \leqslant r \leqslant n-1)$ is nonsingular and elliptic, we can't deduce that $\det A \neq 0$. For example, consider

$$A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 4 & 1 \\ 1 & 4 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

THEOREM 3.4. *Let* $A \in \bar{\mathscr{K}}_n$. *If* A *has no proper principal submatrix that is elliptic, then the positive eigenvalue is the smallest one (in modulus) among the eigenvalues of* A, *and so* $\det A \neq 0$.

Proof. Since $A \in \bar{\mathscr{K}}_n$, by the interlacing theorem we know that any principal submatrix of A is negative semidefinite. This states that the eigenpolynomial of A is given by

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda - a_n,$$

where $a_i \geqslant 0$. If $a_n = 0$, then $p(\lambda) = 0$ has no positive root. This is a contradiction. Therefore, $a_n > 0$ and we have

$$p(\lambda) = -a_n\lambda^n\left[\left(\frac{1}{\lambda}\right)^n - \frac{a_{n-1}}{a_b}\left(\frac{1}{\lambda}\right)^{n-1} - \cdots - \frac{a_1}{a_n}\left(\frac{1}{\lambda}\right) - \frac{1}{a_n}\right],$$

which gives the assertion by Lemma 1.3. ∎

DEFINITION. We call a simple connected graph elliptic of rank r if its adjacency matrix is elliptic and has rank r.

As an application of the previous results, we now give Smith's results (see [4, pp. 403–406] or [5, p. 163, Theorem 6.7]).

THEOREM 3.5. *A simple graph of order n is complete k-partite if and only if it is an elliptic graph of rank k.*

Proof. The "only if" part: Since we may assume that the adjacency matrix of a complete k-partite graph is given by

$$A = \begin{pmatrix} 0 & A_{12} & A_{13} & \cdots & A_{1k} \\ A_{21} & 0 & A_{23} & \cdots & A_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ A_{k1} & A_{k2} & A_{k3} & \cdots & 0 \end{pmatrix}, \tag{2}$$

where A_{ij} is the $l_i \times l_j$ matrix with all entries equal to 1, i, $j = 1, 2, \ldots, k$, and $\sum_{i=1}^{k} l_i = n$, it suffices to prove that A has exactly one positive eigenvalue. By the Perron-Frobenius theorem [3], A has at least one positive eigenvalu. On the other hand, noticing that

$$A = E_n - \begin{pmatrix} E_{l_1} & & 0 \\ & \ddots & \\ 0 & & E_{l_k} \end{pmatrix},$$

where E_i is the $i \times i$ matrix with all entries equal to 1, $i = l_1, l_2, \dots, l_k$, for any real vector $x = (x_1, x_2, \dots, x_n)^T$ satisfying $x^T e = 0$ we have that

$$x^T A x = -\Big[(x_1 + x_2 + \cdots + x_{l_1})^2 + (x_{l_1} + \cdots + x_{l_1 + l_2})^2 + \cdots$$
$$+ (x_{l_1 + \cdots + l_{k-1} + 1} + \cdots + x_{l_1 + \cdots + l_k})^2\Big]$$
$$\leqslant 0,$$

which indicates that A has at most one positive eigenvalue.

The "if" part: Let the adjacency matrix of an elliptic graph be B; then it is readily seen that, by Lemma 2.5, there exists a permutation matrix P such that $P^T B P = A$, where A is given by (2). This implies that such an elliptic graph and a complete k-partite graph are isomorphic. ∎

We would like to thank Professor M. Fiedler for bringing elliptic matrices to our attention. We are also very grateful to the referee for helping us to improve the proof and pointing out some errors in the original version of this paper.

REFERENCES

1 Miroslav Fiedler, Elliptic matrices with zero diagonal, *Linear Algebra Appl.* 197, 198:337–347 (1994).
2 G. Polya and G. Szegö, *Problems and Theorems in Analysis*, Vol. 1, Springer-Verlag, 1972.
3 S. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic, New York, 1979.
4 J. H. Smith, Some properties of the spectrum of a graph, in *Combinatorial Structures and Their Applications* (R. Guy, N. Saur, and J. Schönheim, Eds.), Gordon and Breach, New York, 1970.
5 D. M. Cvetkovic, M. Doob, and H. Sachs, *Spectra of Graphs–Theory and Application*, Berlin, 1980.

Received 8 May 1995; final manuscript accepted 24 March 1996