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Note

The numbers of spanning trees of the cubic cycle C_N^3 and the quadruple cycle C_N^4

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Abstract

The numbers of spanning trees of the cubic cycle C_n^3 and the quadruple cycle C_n^4 are considered in this paper. Two recursive relations are obtained. When we use our approach to consider the square cycle C_n^2, the proof is simpler than the previous ones. Furthermore, we may deal with the general case with the aid of the ideas and techniques in this paper.

1. Introduction and notation

For the cycle graph $G = C_n^p$, i.e., the graph $G = C_n^p$ has points labelled as $0, 1, 2, \ldots, n-1$ and each point i, $0 \leqslant i \leqslant n-1$, is adjacent to the points $i+1, i+p$ (mod n), respectively, we denote by $T(C_n^p)$ the number of spanning trees (the complexity) of C_n^p. The formula for $T(C_n^2)$ was originally conjectured by Bedrosian and subsequently proved by Kleitman and Golden [5]. Without knowledge of Kleitman and Golden [5], the same formula was also conjectured by Boesch and Wang [2]. Different proofs of the formula can be seen in [1, 3, 6], in which it is given as follows:

$$T(C_n^2) = n F_n^2,$$

where F_n is the Fibonacci number defined by the recursive relation

$$F_n = F_{n-1} + F_{n-2}, \quad n = 2, 3, \ldots,$$

with the initial condition $F_0 = 0$, $F_1 = 1$. The present paper provides the formulas for $T(C_n^3)$ and for $T(C_n^4)$. Furthermore, one can consider the general case using the ideas and techniques in this paper.

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2. Some basic results

Lemma 1 (*Biggs* [4]).

$$T(C_n^p) = \frac{1}{n}\prod_{j=1}^{n-1}(4 - \varepsilon^j - \varepsilon^{-j} - \varepsilon^{pj} - \varepsilon^{-pj}),$$

where ε^{-j} *is the conjugate of* ε^j, $\varepsilon = e^{2\pi i/n}$, $1 \leqslant p \leqslant [n/2]$.

Lemma 2. *Let*

$$f_p(x) = x^{2p-2} + 2x^{2p-3} + \cdots + (p-1)x^p + (p+1)x^{p-1} + (p-1)x^{p-2} + \cdots + 2x + 1.$$

Then we have the following determinantal expression of $T(C_n^p)$:

$$T(C_n^p) = \frac{1}{n}A_n = (-1)^{(p-1)(n-1)}\frac{n}{f_p(1)}|-\bar{A}_p^n + I|,$$

where $\bar{A}_p$ *is the companion matrix of* $f_p(x)$, $p = 1, 2, \ldots, [n/2]$, *that is,*

$$\bar{A}_p = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & -2 \\ & & & \cdots & & \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & -(p-1) \\ 0 & 0 & 0 & \cdots & 0 & -(p+1) \\ 0 & 0 & 0 & \cdots & 0 & -(p-1) \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix}_{(2p-2)\times(2p-2)}$$

and I *is the identity matrix of order* $2(p-1)$.

Proof. Because we have

$$A_n = \prod_{j=1}^{n-1}(4 - \varepsilon^j - \varepsilon^{-j} - \varepsilon^{pj} - \varepsilon^{-pj})$$
$$= (-1)^{(n-1)}\prod_{j=1}^{n-1}\varepsilon^{-pj}\prod_{j=1}^{n-1}(\varepsilon^j - 1)^2\prod_{j=1}^{n-1}(\varepsilon^{(2p-2)j} + 2\varepsilon^{(2p-3)j} + \cdots + (p-1)\varepsilon^{pj} + (p+1)\varepsilon^{(p-1)j} + (p-1)\varepsilon^{(p-2)j} + \cdots + 2\varepsilon^j + 1),$$

and that

$$f_p(x) = |xI - \bar{A}_p| = x^{2p-2} + 2x^{2p-3} + \cdots + (p-1)x^p + (p+1)x^{p-1} + (p-1)x^{p-2} + \cdots + 1,$$

it yields

$$\prod_{j=1}^{n-1}(\varepsilon^{(2p-2)j}+2\varepsilon^{(2p-3)j}+\cdots+(p-1)\varepsilon^{pj}$$
$$+(p+1)\varepsilon^{(p-1)j}+(p-1)\varepsilon^{(p-2)j}+\cdots+2\varepsilon^{j}+1)$$
$$=\prod_{j=1}^{n-1}|\varepsilon^{j}I-\bar{A}_p|.$$

Now,

$$|I-\bar{A}_p|=f_p(1),$$
$$\prod_{j=1}^{n-1}(\varepsilon^{j}-1)=-n,$$
$$\prod_{j=1}^{n-1}\varepsilon^{-pj}=(-1)^{-p(n-1)},$$

we have

$$A_n=\frac{(-1)^{(p-1)(n-1)}}{|I-\bar{A}_p|}\prod_{j=1}^{n-1}(\varepsilon^{j}-1)^2\prod_{j=1}^{n}|\varepsilon^{j}I-\bar{A}_p|$$
$$=(-1)^{(p-1)(n-1)}\,n^2\,\frac{|I-\bar{A}_p^{n}|}{f_p(1)}.$$

This completes the proof of Lemma 2. □

3. The main results

Case (a): $p = 3$.

Theorem 3. *The following relation holds:*

$$T(C_n^3)=\frac{1}{n}A_n=na_n^2,$$

where a_n satisfies the recursive relation

$$a_n=\sqrt{2}(a_{n-1}+a_{n-3})-a_{n-4}, \qquad (1)$$

with the initial condition $a_1 = 1$, $a_2 = 2\sqrt{2}$, $a_3 = 5$, $a_4 = 5\sqrt{2}$ (they are easily obtained by Lemma 2).

Proof. By virtue of Lemma 2, we have

$$A_n=n^2\prod_{j=1}^{n-1}f_3(\varepsilon^{j}).$$

This deduces, by letting that $f_3(x) = (x - \alpha)(x - \alpha^{-1})(x - \bar{\alpha})(x - \bar{\alpha}^{-1})$ (from the expression of $f_3(x)$, such an assumption is feasible), the following:

$$\begin{aligned} a_n^2 &= \prod_{j=1}^{n-1} (\varepsilon^j - \alpha)(\varepsilon^j - \alpha^{-1})(\varepsilon^j - \bar{\alpha})(\varepsilon^j - \bar{\alpha}^{-1}) \\ &= \frac{(1 - \alpha^n)(1 - \alpha^{-n})(1 - \bar{\alpha}^n)(1 - \bar{\alpha}^{-n})}{(1 - \alpha)(1 - \alpha^{-1})(1 - \bar{\alpha})(1 - \bar{\alpha}^{-1})} \\ &= \left[\frac{(1 - \alpha^n)(1 - \bar{\alpha}^n)}{\sqrt{10}|\alpha|^n}\right]^2, \end{aligned}$$

where $(1 - \alpha)(1 - \alpha^{-1})(1 - \bar{\alpha})(1 - \bar{\alpha}^{-1}) = f_3(1) = 10$. Therefore, we can readily check that

$$a_n = \frac{1}{\sqrt{10}}\left[|\alpha|^n + \left(\frac{1}{|\alpha|}\right)^n - \left(\frac{\alpha}{|\alpha|}\right)^n - \left(\frac{\bar{\alpha}}{|\alpha|}\right)^n\right]. \tag{2}$$

Now, we are to verify that a_n is the solution of difference equation (1). According to (2), we know that a_n is a solution of a difference equation of order 4. That is,

$$a_n + aa_{n-1} + ba_{n-2} + ca_{n-3} + da_{n-4} = 0.$$

Since seen by (2) that $|\alpha|$, $1/|\alpha|$, $\bar{\alpha}/|\alpha|$ are eigenvalues of a_n, we have readily by Vita's theorem and the expression of $f_3(x)$ that $a = c$, $d = 1$. Hence,

$$a_n + a(a_{n-1} + a_{n-3}) + ba_{n-2} + a_{n-4} = 0.$$

With the help of the initial condition and noting that $a_5 = 13$, $a_6 = 16\sqrt{2}$ (by Lemma 2), it gives the equation

$$\begin{cases} 7\sqrt{2a} + 5b = -14, \\ 18a + 5\sqrt{2b} = -18\sqrt{2}. \end{cases}$$

This implies that $a = -\sqrt{2}$, $b = 0$. Therefore, we have the recursive relation as follows:

$$a_n = \sqrt{2}(a_{n-1} + a_{n-3}) - a_{n-4}.$$

The proof is completed.

Case (b): $p = 4$.

Theorem 4.

$$T(C_n^4) = \frac{1}{n} A_n = na_n^2,$$

where a_n satisfies the recursive relation,

$$a_n = a_{n-1} + a_{n-3} + 3a_{n-4} - a_{n-5} - a_{n-7} - a_{n-8},$$

with the initial condition

$$a_1 = 1, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 1, \quad a_5 = 4, \quad a_6 = 8, \quad a_7 = 13, \quad a_8 = 17.$$

The a_n are also easily obtained by Lemma 2, $n = 1, 2, \ldots, 8$, which may be easily calculated by a computer.

Proof. Repeating the procedure of proving Theorem 3, and letting

$$\begin{aligned} f_4(x) &= x^6 + 2x^5 + 3x^4 + 5x^3 + 3x^2 + 2x + 1 \\ &= (x-\alpha)(x-\alpha^{-1})(x-\beta)(x-\beta^{-1})(x-\gamma)(x-\gamma^{-1}), \end{aligned}$$

we get

$$\begin{aligned} a_n^2 &= (-1)^{3(n-1)} \frac{(1-\alpha^n)(1-\alpha^{-n})(1-\beta^n)(1-\beta^{-n})(1-\gamma^n)(1-\gamma^{-n})}{(1-\alpha)(1-\alpha^{-1})(1-\beta)(1-\beta^{-1})(1-\gamma)(1-\gamma^{-1})} \\ &= \frac{(1-\alpha^n)^2(1-\beta^n)^2(1-\gamma^n)^2}{(-\alpha\beta\gamma)^n f_4(1)}. \end{aligned}$$

Let

$$a_n = \frac{1}{\sqrt{f_4(1)}}(\alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n + \alpha_5^n + \alpha_6^n + \alpha_7^n + \alpha_8^n).$$

Then, as the case $p = 3$, we may suppose that

$$a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_8 a_{n-8}.$$

This gives the equation (the coefficient matrix is a Toeplitz matrix)

$$\begin{pmatrix} a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \\ a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{15} & a_{14} & a_{13} & a_{12} & a_{11} & a_{10} & a_9 & a_8 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_8 \end{pmatrix} = \begin{pmatrix} a_9 \\ a_{10} \\ \vdots \\ a_{16} \end{pmatrix}, \tag{3}$$

where $a_i, 1 \leqslant i \leqslant 8$, are given by Theorem 4. By Lemma 2 we obtain easily the accurate values of a_i, $i = 9, \ldots, 16$, i.e.,

$$a_9 = 34, \quad a_{10} = 64, \quad a_{11} = 149, \quad a_{12} = 176, \quad a_{13} = 313,$$

$$a_{14} = 559, \quad a_{15} = 968, \quad a_{16} = 1649.$$

By (3), we obtain easily that

$$b_1 = 1, \quad b_2 = 0, \quad b_3 = 1, \quad b_4 = 3, \quad b_5 = -1, \quad b_6 = 0,$$

$$b_7 = -1, \quad b_8 = -1.$$

Therefore,

$$a_n = a_{n-1} + a_{n-3} + 3a_{n-4} - a_{n-5} - a_{n-7} - a_{n-8}.$$

The proof is completed. □

As a by-product, we now consider $T(C_n^2)$. Since

$$A_n = n^2 a_n^2 = (-1)^{n-1} \frac{(1-a^n)(1-1/a^n)}{(1-a)(1-1/a)},$$

we may suppose that

$$a_n = a a_{n-1} + b a_{n-2}.$$

By Lemma 2, we obtain $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3$. Therefore, we have

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

which implies that $a = b = 1$.

Corollary 5.

$$T(C_n^2) = n a_n^2,$$

where

$$a_n = a_{n-1} + a_{n-2}, \quad n = 2, 3, \ldots,$$

with initial condition $a_0 = 0, a_1 = 1$.

Corollary 5 is the conjecture of Boesch and Wang [2].

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