Discrete Mathematics 338 (2015) 1883–1906

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The formulas for the number of spanning trees in circulant graphs[☆]

Min Li[a], Zhibing Chen[a], Xiaoqing Ruan[a], Xuerong Yong[b,*]

[a] College of Mathematics and Computational Science, Shenzhen University, Shenzhen, China
[b] Department of Mathematical Sciences, University of Puerto Rico, Mayaguez, USA

ARTICLE INFO

Article history:
Received 2 July 2014
Received in revised form 25 April 2015
Accepted 27 April 2015

Keywords:
Circulant graphs
Spanning trees
Asymptotic limit
Chebyshev polynomials

ABSTRACT

The circulant graph $C_n^{s_1,s_2,\ldots,s_t}$ is the $2t$ regular graph with n vertices labeled $0, 1, 2, \ldots, n-1$, where each vertex i has the $2t$ neighbors $i \pm s_1, i \pm s_2, \ldots, i \pm s_t$ in which all the operations are modulo n. Golin et al. (2010) derive several closed integral formulas for the asymptotic limit

$$\lim_{n\to\infty} T\left(C_n^{s_1,s_2,\ldots,s_k,\lfloor \frac{n}{d_1}\rfloor+e_1,\lfloor \frac{n}{d_2}\rfloor+e_2,\ldots,\lfloor \frac{n}{d_l}\rfloor+e_l}\right)^{\frac{1}{n}},$$

as a function of s_i, d_j and e_k, where $T(G)$ is the number of spanning trees in graph G.

In this paper we derive simple and explicit formulas for the number of spanning trees in circulant graphs $C_{pn}^{1,a_1n,a_2n,\ldots,a_ln}$. Following from the formulas we show that

$$\lim_{n\to\infty} T\left(C_{pn}^{1,a_1n,a_2n,\ldots,a_ln}\right)^{\frac{1}{n}} = \prod_{t=0}^{k-1}\left(\sqrt{1+\sum_{i=1}^{l}\sin^2\frac{\pi a_i t}{p}}+\sqrt{\sum_{i=1}^{l}\sin^2\frac{\pi a_i t}{p}}\right)^{\frac{2p}{k}},$$

where $k = \mathrm{lcm}(\frac{p}{a_1}, \frac{p}{a_2}, \ldots, \frac{p}{a_l})$, and lcm denotes the least common multiple. The asymptotic limit represents the average growth rate of the number of spanning trees.

The research is continuation of the previous work (Golin et al., 2010; Zhang et al., 2000; Zhang et al., 2005).

Published by Elsevier B.V.

1. Introduction

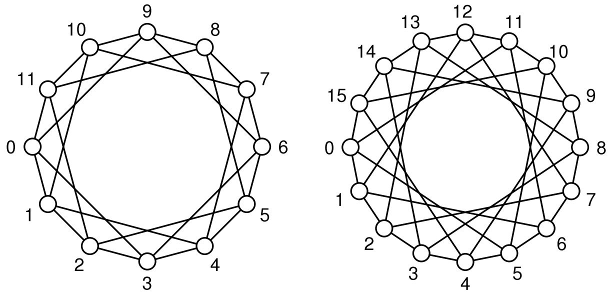
Throughout this paper, the graphs are allowed to contain multiple edges and self-loops unless otherwise specified.

Let G be a connected graph on n vertices. *A spanning tree* in G is a tree having the same vertex set as G and its edge set is a subset of the edge set of G. *An oriented spanning tree* in a directed graph $\vec{G}$ is a rooted tree with the same vertex set as $\vec{G}$,

☆ The research was supported in part by Shenzhen University. The fourth author's work was also supported by DIMACS and the University of Puerto Rico at Mayaguez.

* Corresponding author.
E-mail addresses: lminyy@sina.com (M. Li), chenzb@szu.edu.cn (Z. Chen), ruanxq@szu.edu.cn (X. Ruan), xryong@dimacs.rutgers.edu (X. Yong).

http://dx.doi.org/10.1016/j.disc.2015.04.025
0012-365X/Published by Elsevier B.V.



(a) $C_{12}^{1,3}$. (b) $C_{16}^{1,5}$.

Fig. 1. Two special circulant graphs.

i.e., there is a specified root node and paths from it to every vertex of $\vec{G}$. The number of spanning trees has a long research history because it is interesting from a combinatorial perspective and also arises in several application problems. An old motivation is that the number of spanning trees characterizes basically the reliability of a network in the presence of line fault and counting the number of spanning trees is important in designing electrical circuits [7]. Another related discovery is that the resistance distance between two nodes in a network can be expressed by the number of spanning trees and therefore computation of the effective electric resistance involves calculation/estimation of the number of spanning trees in a given network (see e.g., Theorem 7–4 in [16], or [1,17]).

Given the adjacency matrix of a graph G, *Kirchoff's matrix tree theorem* [11] gives a closed formula for calculating the number of spanning trees. However, for a given general graph it is not easy to count the exact number (or to get a good estimation) from the theorem and therefore special graphs have been considered extensively because it turns out to be possible to derive explicit formulas for them. The real problem, then, is to calculate the number of spanning trees of special graphs in particular parameterized classes. The class of circulant graphs to be considered here is one of the classes of graphs that have been received much attention in the last decades.

Circulant graphs can be defined in different ways [2,4,10]. Here it is convenient to introduce them and the values to be counted as described in [10]. Let $s_1 < s_2 < \cdots < s_k \le \lfloor \frac{n}{2} \rfloor$ be given positive integers. The circulant graph on n vertices and jumps $s_1, s_2, \ldots, s_k$ is defined by

$$C_n^{s_1,s_2,\ldots,s_k} = (V, E)$$

where $V = \{0, 1, 2, \ldots, n-1\}$, and

$$E = \bigcup_{i=0}^{n-1} \{ (i, i+s_1),\ (i, i+s_2), \ldots, (i, i+s_k)\}^{1}$$

where all of the additions are done $\mod n$. That is, each node is connected to the nodes that are jumps s_j away from it, for $j = 1, 2, \ldots, k$. Similarly, the directed circulant graph on n vertices,[2] $\vec{C}_n^{s_1,s_2,\ldots,s_k}$, has the same vertex set but $s_k < n$ and

$$E = \bigcup_{i=0}^{n-1} \{ (i, i+s_1),\ (i, i+s_2), \ldots, (i, i+s_k)\}$$

where an edge is *directed* from each i to the nodes s_j ahead of it, for $j = 1, 2, \ldots, k$. To avoid confusion, we emphasize that, since we are allowing multiple edges in our graphs, $C_n^{s_1,s_2,\ldots,s_k}$ is *always* $2k$-regular and $\vec{C}_n^{s_1,s_2,\ldots,s_k}$ is *always* k-regular. For example, in our notation, $C_{2n}^{1,n}$ is the 4-regular graph with $2n$ vertices such that each vertex i is connected by *one* edge to each of $(i-1) \bmod 2n$ and $(i+1) \bmod 2n$ and by *two* edges to $(i+n) \bmod 2n$. Our techniques would, with slight technical modifications, also permit analyzing graphs in which multiple edges are *not* allowed, e.g., the *Mobius ladder* M_{2n}. This is the 3 regular graph with $2n$ vertices such that each vertex i is connected by one edge to each of $(i-1) \bmod 2n$, $(i+1) \bmod 2n$ and $(i+n) \bmod 2n$. The reason that we do not explicitly analyze such graphs is that such an analysis would require rewriting all of our theorems a second time to deal with these special instances without introducing any new interesting techniques. Two undirected circulant graphs are given in Fig. 1.

Let $T(X)$ stand for the number of spanning trees in a (directed or undirected) graph X. For any fixed integers $1 \le s_1 \le s_2 \le \cdots \le s_k$, it was shown in [19] that, in the case of *directed circulant graphs*,

$$\lim_{n\to\infty} T(\vec{C}_n^{s_1,s_2,\ldots,s_k})^{\frac{1}{n}} = \lim_{n\to\infty} \frac{T(\vec{C}_{n+1}^{s_1,s_2,\ldots,s_k})}{T(\vec{C}_n^{s_1,s_2,\ldots,s_k})} = k,$$

[1] We should mention that the general concept of a set does not allow for multiple copies of an element. Here multiple edges can appear in the graph when performing addition modulo n on the stage of interpreting the elements of E.

[2] In this paper we do not discuss directed graphs. We talk about them for completeness and the reason we address some of their results is that there is an open problem on them, which is emphasized in the Concluding Remarks.

where k is the degree of each vertex of $\vec{C}_{n+1}^{s_1,s_2,\ldots,s_k}$. However, for *undirected graphs*, the asymptotic limits was shown to be dependent on k and all the s_i [10], and

$$\lim_{n\to\infty} T(C_n^{s_1,s_2,\ldots,s_k})^{\frac{1}{n}} = \lim_{n\to\infty} \frac{T(C_{n+1}^{s_1,s_2,\ldots,s_k})}{T(C_n^{s_1,s_2,\ldots,s_k})} = 4\exp\left[\int_0^1 \ln\left(\sum_{i=1}^{k} \sin^2 \pi s_i x\right) dx\right]. \tag{1}$$

More generally, in [10] Golin et al. derive a closed integral formula for the asymptotic limit

$$\lim_{n\to\infty} T\left(C_n^{s_1,s_2,\ldots,s_k,\lfloor \frac{n}{d_1}\rfloor+e_1,\lfloor \frac{n}{d_2}\rfloor+e_2,\ldots,\lfloor \frac{n}{d_l}\rfloor+e_l}\right)^{\frac{1}{n}},$$

as a function of s_i, d_j and e_k. (We do not write out the formula because it is complicated.)

Note that the limits represent the *average growth rate* of the number of spanning trees in directed and undirected circulant graphs, respectively. It is known that the asymptotic limits for grids and tori turn out to be equal and are also equal to the limit for the circulant graphs with non-constant jumps [5,8,10].[3] For general recursive graphs Lyons [15] has developed a more general technique for deriving the asymptotics of the number of spanning trees. His techniques can be used to derive the asymptotics of fixed-jump circulants, but do not seem to be usable to derive the asymptotics when the jumps depend on n. The number of spanning trees in directed circulants with non-constant jumps was studied in [6]. For circulant graphs with non-constant jumps $C_{pn}^{s_1,\ldots,s_k,a_1n,\ldots,a_ln}$, in [10] Golin et al. derive an asymptotic closed integral formula for the number of spanning trees. Recent results on calculating the asymptotic (maximum) number of spanning trees in circulant graphs with both fixed-jumps and the jumps depending linearly upon n can be seen in [10,12,14]. We would like to address that when graphs have jumps with sizes depending on n the situation becomes complicated. For instance, from [10] we observe that for $q = 0, 1, 2,$

$$\lim_{\substack{n\to\infty \\ n \bmod 3=q}} T\left(C_n^{1,\lfloor \frac{n}{3}\rfloor}\right) = c_q$$

where the c_q are three different constants.

In this paper we consider the circulant graphs $C_{pn}^{1,a_1n,a_2n,\ldots,a_ln}$, continuing the previous work [10,20,21]. In Section 2 we derive simple and explicit formulas for the number of spanning trees of the graphs and their asymptotics. Following from the formulas derived we obtain the value for the limiting asymptotics of the number of spanning trees in circulant graphs $C_{pn}^{1,a_1n,a_2n,\ldots,a_ln}$.

To derive our explicit formulas we need to recall the fundamentals for the number of spanning trees and various standard properties of the second kind Chebyshev polynomials. Our basic idea actually is an extended use of these polynomials, compared with the ones applied in [18,21]. First, making use of Kirchoff's Matrix Tree Theorem it is easily shown that (e.g., [2,10])

$$T(C_n^{s_1,s_2,\ldots,s_k}) = \frac{1}{n}\prod_{j=1}^{n-1}\left(2k - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{n}\right). \tag{2}$$

Then, for convenience, we write out the useful properties of the Chebyshev polynomials copied from [3,21]. For a positive integer m, the Chebyshev polynomials of the first kind are defined as $T_m(x) = \cos(m \arccos x)$, and the second kind are

$$U_{m-1}(x) = \frac{1}{m}\frac{d}{dx}T_m(x) = \frac{\sin(m\arccos x)}{\sin(\arccos x)}.$$

It is easily verified that $U_m(x) - 2xU_{m-1}(x) + U_{m-2}(x) = 0$, and

$$U_m(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^{m+1} - \left(x-\sqrt{x^2-1}\right)^{m+1}\right], \tag{3}$$

where the identity is true for all complex x (at points $x = \pm 1$ take the limits), and by which we can easily obtain that

$$U_{m-1}(x) = 2^{m-1}\prod_{j=1}^{m-1}\left(x - \cos\frac{j\pi}{m}\right), \qquad U_{m-1}^2(x) = 4^{m-1}\prod_{j=1}^{m-1}\left(x^2 - \cos^2\frac{j\pi}{m}\right).$$

Finally, manipulating the above formulas yields the extremely useful formula in our consideration:

$$U_{m-1}^2\left(\sqrt{\frac{x+2}{4}}\right) = \prod_{j=1}^{m-1}\left(x - 2\cos\frac{2\pi j}{m}\right). \tag{4}$$

2. Simple and explicit spanning tree formulas

The following Theorem A is Theorem 3 of [10].

[3] A jump s_j is called non-constant if it is a function of n, the number of vertices of graph. While examining the structure of non-constant jump circulant graphs, it was conjectured in [9] that the asymptotics of the number of spanning trees of the $m \times n$ tori and grids and the circulant graphs $C_{mn}^{1,n}$ would be the same.

Theorem A. *Let* $1 \le s_1 \le \cdots \le s_k$, p *and* $1 \le a_1 \le \cdots \le a_l < p$ *be integers. Then*

$$\lim_{n\to\infty} T(C_{pn}^{s_1,\ldots,s_k,a_1n,\ldots,a_ln})^{\frac{1}{n}} = 4^p \exp\left[\sum_{t=0}^{p-1}\int_0^1 \ln\left(\sum_{i=1}^{k}\sin^2 \pi s_i x + \sum_{i=1}^{l}\sin^2\frac{\pi a_i t}{p}\right)dx\right].$$

Note that if $p = 1$, then $l = 0$ and Theorem A reduces to (1). To derive our formulas, we use the idea introduced in [10] and establish the following lemma.

Lemma 1. *Let* $0 \le s_1 \le \cdots \le s_k$, p *and* $0 \le a_1 \le \cdots \le a_l < p$ *be integers. Then*

$$\begin{aligned}
T\left(C_{pn}^{s_1,\ldots,s_k,a_1n,\ldots,a_ln}\right) &= \frac{1}{pn}\prod_{j=1}^{pn-1}\left[2(k+l) - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn} - 2\sum_{i=1}^{l}\cos\frac{2\pi a_i n j}{pn}\right] \\
&= \frac{1}{pn}\prod_{t=1}^{\lfloor \frac{p}{2}\rfloor}\left(\prod_{\substack{j=1\\ j \bmod p = t, p-t}}^{pn-1}\left[2(k+l) - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn} - 2\sum_{i=1}^{l}\cos\frac{2\pi a_i t}{p}\right]\right) \\
&\quad \times \prod_{j'=1}^{n-1}\left[2k - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j'}{n}\right].
\end{aligned}$$

Proof. From Formula (2) we see that

$$\begin{aligned}
T\left(C_{pn}^{s_1,\ldots,s_k,a_1n,\ldots,a_ln}\right) &= \frac{1}{pn}\prod_{j=1}^{pn-1}\left[2(k+l) - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn} - 2\sum_{i=1}^{l}\cos\frac{2\pi a_i n j}{pn}\right] \\
&= \frac{1}{pn}\prod_{\substack{j=1\\ j \bmod p \ne 0}}^{pn-1}\left[2(k+l) - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn} - 2\sum_{i=1}^{l}\cos\frac{2\pi a_i n j}{pn}\right] \\
&\quad \times \prod_{\substack{j=1\\ j \bmod p = 0}}^{pn-1}\left[2(k+l) - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn} - 2\sum_{i=1}^{l}\cos\frac{2\pi a_i n j}{pn}\right].
\end{aligned}$$

Note that if $j = mp$ for some integer m then $\cos\frac{2\pi a_i nj}{pn} = \cos\frac{2\pi a_i mp}{p} = 1$, and

$$\prod_{\substack{j=1\\ j \bmod p = 0}}^{pn-1}\left[2(k+l) - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn} - 2\sum_{i=1}^{l}\cos\frac{2\pi a_i n j}{pn}\right] = \prod_{j=1}^{n-1}\left(2k - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{n}\right).$$

Let $0 < t \le \left[\frac{p}{2}\right]$. Noting that if $j = mp + t$ for some integer m then

$$\cos\frac{2\pi a_i nj}{pn} = \cos\frac{2\pi a_i(mp+t)}{p} = \cos\frac{2\pi a_i t}{p},$$

and if $j \bmod p = p - t$ then $j = mp + p - t = p(m+1) - t$ for some integer m, and

$$\cos\frac{2\pi a_i nj}{pn} = \cos\frac{2\pi a_i[p(m+1)-t]}{p} = \cos\frac{2\pi a_i t}{p},$$

by these relations we have

$$\begin{aligned}
&\prod_{\substack{j=1\\ j \bmod p \ne 0}}^{pn-1}\left[2(k+l) - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn} - 2\sum_{i=1}^{l}\cos\frac{2\pi a_i n j}{pn}\right] \\
&= \prod_{t=1}^{p-1}\left(\prod_{\substack{j=1\\ j \bmod p = t}}^{pn-1}\left[2(k+l) - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn} - 2\sum_{i=1}^{l}\cos\frac{2\pi a_i n j}{pn}\right]\right) \\
&= \prod_{t'=1}^{\lfloor \frac{p}{2}\rfloor}\left(\prod_{\substack{j=1\\ j \bmod p = t', p-t'}}^{pn-1}\left[2(k+l) - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn} - 2\sum_{i=1}^{l}\cos\frac{2\pi a_i t'}{p}\right]\right).
\end{aligned}$$

Consequently,

$$T\left(C_{pn}^{s_1,\dots,s_k,a_1n,\dots,a_ln}\right) = \frac{1}{pn}\prod_{t=1}^{\lfloor \frac{p}{2}\rfloor}\left(\prod_{\substack{j=1\\ j \bmod p=t,p-t}}^{pn-1}\left[2(k+l)-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn}-2\sum_{i=1}^{l}\cos\frac{2\pi a_i t}{p}\right]\right)$$
$$\times\prod_{j'=1}^{n-1}\left[2k-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j'}{n}\right].$$

The proof is completed.

For some special circulant graphs with fixed and non-fixed jumps, in [20,21] Zhang et al. derive several formulas for the number of spanning trees. For more general circulants, making use of Lemma 1 it is possible to derive *simple and explicit* formulas for the numbers of spanning trees and obtain their corresponding asymptotic properties. Below, as examples, we derive the spanning tree formulas for more general circulants (We derive the cases from (a) to (d). The derivations of the formulas from (e) to (o) are given in Appendix at the end of the paper), where we consider the cases $s_1 = 1, s_j = 0, j > 1$ and $\frac{p}{a_i} \in \{2, 3, 4, 6\}$. Our idea is, for different values of $\sum_{i=1}^{l}\cos\frac{2\pi a_i j}{p}$ in Lemma 1, to write the spanning tree formula as the products of appropriate values of Chebyshev polynomials, where we first plug the value of p into the corresponding formula in Lemma 1 and then represent the products in terms of appropriate values of the Chebyshev polynomials by making use of Formulas (4) and then (3).

(I) The case $T\left(C_{pn}^{1,bn}\right)$, where b, p are any positive integers satisfying $\frac{p}{b} \in \{2, 3, 4, 6\}$. We need to derive the formulas separately.

(a) If $\frac{p}{b} = 2$, then plugging $p = 2b$ into the formula in Lemma 1 and then representing the products in terms of appropriate values of the Chebyshev polynomials by using Formulas (3) and (4) yield

$$\begin{aligned}
T\left(C_{pn}^{1,bn}\right) &= \frac{1}{2bn}\prod_{t=1}^{\lfloor\frac{2}{2}\rfloor}\left(\prod_{\substack{j=1\\ j \bmod 2=t,2-t}}^{2bn-1}\left[4-2\cos\frac{2\pi j}{2bn}-2\cos\frac{2\pi t}{2}\right]\right)\times\prod_{j=1}^{bn-1}\left[2-2\cos\frac{2\pi j}{bn}\right]\\
&= \frac{1}{2bn}\prod_{\substack{j=1\\ j \bmod 2=1}}^{2bn-1}\left[4-2\cos\frac{2\pi j}{2bn}-2\cos\frac{2\pi}{2}\right]\times\prod_{j=1}^{bn-1}\left[2-2\cos\frac{2\pi j}{bn}\right]\\
&= \frac{1}{2bn}\prod_{j=1}^{2bn-1}\left[6-2\cos\frac{2\pi j}{2bn}\right]\frac{1}{\prod\limits_{\substack{j=1\\ j \bmod 2=0}}^{2bn-1}\left[6-2\cos\frac{2\pi j}{2bn}\right]}\times\prod_{j=1}^{bn-1}\left[2-2\cos\frac{2\pi j}{bn}\right]\\
&= \frac{1}{2bn}\prod_{j=1}^{2bn-1}\left[6-2\cos\frac{2\pi j}{2bn}\right]\prod_{j=1}^{bn-1}\frac{2-2\cos\frac{2\pi j}{bn}}{6-2\cos\frac{2\pi j}{bn}}\\
&= \frac{1}{2bn}U_{2bn-1}^2\left(\sqrt{\frac{6+2}{4}}\right)\frac{U_{bn-1}^2\left(\sqrt{\frac{2+2}{4}}\right)}{U_{bn-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}\\
&= \frac{1}{2bn}U_{2bn-1}^2\left(\sqrt{2}\right)\frac{U_{bn-1}^2(1)}{U_{bn-1}^2\left(\sqrt{2}\right)}\\
&= \frac{1}{2bn}\left[\frac{1}{2\sqrt{\left(\sqrt{2}\right)^2-1}}\left(\sqrt{2}+\sqrt{\left(\sqrt{2}\right)^2-1}\right)^{2bn}-\left(\sqrt{2}-\sqrt{\left(\sqrt{2}\right)^2-1}\right)^{2bn}\right]^2\\
&\quad\times\frac{(bn)^2}{\left[\frac{1}{2\sqrt{\left(\sqrt{2}\right)^2-1}}\left(\sqrt{2}+\sqrt{\left(\sqrt{2}\right)^2-1}\right)^{bn}-\left(\sqrt{2}-\sqrt{\left(\sqrt{2}\right)^2-1}\right)^{bn}\right]^2}\\
&= \frac{bn}{2}\left[\frac{\left(\sqrt{2}+1\right)^{2bn}-\left(\sqrt{2}-1\right)^{2bn}}{\left(\sqrt{2}+1\right)^{bn}-\left(\sqrt{2}-1\right)^{bn}}\right]^2
\end{aligned}$$

$$= \frac{bn}{2}\left[\frac{\left[\left(\sqrt{2}+1\right)^{bn}+\left(\sqrt{2}-1\right)^{bn}\right]\left[\left(\sqrt{2}+1\right)^{bn}-\left(\sqrt{2}-1\right)^{bn}\right]}{\left(\sqrt{2}+1\right)^{bn}-\left(\sqrt{2}-1\right)^{bn}}\right]^2$$
$$= \frac{bn}{2}\left[\left(\sqrt{2}+1\right)^{bn}+\left(\sqrt{2}-1\right)^{bn}\right]^2$$
$$= \frac{pn}{2^2}\left[\left(\sqrt{2}+1\right)^{\frac{p}{2}n}+\left(\sqrt{2}-1\right)^{\frac{p}{2}n}\right]^2,$$

and so the asymptotics (which represents the average growth rate of number of spanning trees of $C_{pn}^{1,bn}$) is given by

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{2}n}\right)} = \left(\sqrt{2}+1\right)^p.$$

(b) If $\frac{p}{b}=3$, then repeating the same procedure as we used in (a) and applying Formulas (4) and (3) yield

$$T\left(C_{pn}^{1,bn}\right) = \frac{1}{3bn}\prod_{t=1}^{\lfloor\frac{3}{2}\rfloor}\left(\prod_{\substack{j=1\\ j \bmod 3=t,3-t}}^{3bn-1}\left[4-2\cos\frac{2\pi j}{3bn}-2\cos\frac{2\pi t}{3}\right]\right)\times\prod_{j=1}^{bn-1}\left[2-2\cos\frac{2\pi j}{bn}\right]$$
$$= \frac{1}{3bn}\prod_{\substack{j=1\\ j \bmod 3=1,3-1}}^{3bn-1}\left[4-2\cos\frac{2\pi j}{3bn}-2\cos\frac{2\pi}{3}\right]\times\prod_{j=1}^{bn-1}\left[2-2\cos\frac{2\pi j}{bn}\right]$$
$$= \frac{1}{3bn}\prod_{\substack{j=1\\ j \bmod 3=1,2}}^{3bn-1}\left[5-2\cos\frac{2\pi j}{3bn}\right]\times\prod_{j=1}^{bn-1}\left[2-2\cos\frac{2\pi j}{bn}\right]$$
$$= \frac{1}{3bn}\prod_{j=1}^{3bn-1}\left[5-2\cos\frac{2\pi j}{3bn}\right]\frac{1}{\prod\limits_{\substack{j=1\\ j \bmod 3=0}}^{3bn-1}\left[5-2\cos\frac{2\pi j}{3bn}\right]}\times\prod_{j=1}^{bn-1}\left[2-2\cos\frac{2\pi j}{bn}\right]$$
$$= \frac{1}{3bn}\prod_{j=1}^{3bn-1}\left[5-2\cos\frac{2\pi j}{3bn}\right]\prod_{j=1}^{bn-1}\frac{2-2\cos\frac{2\pi j}{bn}}{5-2\cos\frac{2\pi j}{bn}}$$
$$= \frac{1}{3bn}U_{3bn-1}^2\left(\sqrt{\frac{5+2}{4}}\right)\frac{U_{bn-1}^2\left(\sqrt{\frac{2+2}{4}}\right)}{U_{bn-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}$$
$$= \frac{1}{3bn}U_{3bn-1}^2\left(\sqrt{\frac{7}{4}}\right)\frac{U_{bn-1}^2(1)}{U_{bn-1}^2\left(\sqrt{\frac{7}{4}}\right)}$$
$$= \frac{1}{3bn}\left[\frac{1}{2\sqrt{\left(\sqrt{\frac{7}{4}}\right)^2-1}}\left(\sqrt{\frac{7}{4}}+\sqrt{\left(\sqrt{\frac{7}{4}}\right)^2-1}\right)^{3bn}-\left(\sqrt{\frac{7}{4}}-\sqrt{\left(\sqrt{\frac{7}{4}}\right)^2-1}\right)^{3bn}\right]^2$$
$$\times\frac{(bn)^2}{\left[\frac{1}{2\sqrt{\left(\sqrt{\frac{7}{4}}\right)^2-1}}\left(\sqrt{\frac{7}{4}}+\sqrt{\left(\sqrt{\frac{7}{4}}\right)^2-1}\right)^{bn}-\left(\sqrt{\frac{7}{4}}-\sqrt{\left(\sqrt{\frac{7}{4}}\right)^2-1}\right)^{bn}\right]^2}$$
$$= \frac{bn}{3}\left[\frac{\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{3bn}-\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{3bn}}{\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{bn}-\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{bn}}\right]^2$$

$$= \frac{bn}{3}\left[\frac{\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2bn}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2bn}+1\right]\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{bn}-\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{bn}\right]}{\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{bn}-\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{bn}}\right]^2$$
$$= \frac{bn}{3}\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2bn}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2bn}+1\right]^2$$
$$= \frac{pn}{3^2}\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{2p}{3}n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{2p}{3}n}+1\right]^2,$$

and so the average growth rate is

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{3}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{3}n}\right)} = \left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{4p}{3}}.$$

(c) If $\frac{p}{b} = 4$, repeating same operations as we did in (a) and (b), and applying Formulas (4) and (3) yields

$$T\left(C_{pn}^{1,bn}\right) = \frac{1}{4bn}\prod_{t=1}^{\lfloor\frac{4}{2}\rfloor}\left(\prod_{\substack{j=1\\ j \bmod 4=t,4-t}}^{4bn-1}\left[4-2\cos\frac{2\pi j}{4bn}-2\cos\frac{\pi t}{2}\right]\right)\times\prod_{j=1}^{bn-1}\left[2-2\cos\frac{2\pi j}{bn}\right]$$
$$= \frac{1}{4bn}\prod_{\substack{j=1\\ j \bmod 4=1,4-1}}^{4bn-1}\left[4-2\cos\frac{2\pi j}{4bn}-2\cos\frac{\pi}{2}\right]\prod_{\substack{j=1\\ j \bmod 4=2}}^{4bn-1}\left[4-2\cos\frac{2\pi j}{4bn}-2\cos\frac{2\pi}{2}\right]\prod_{j=1}^{bn-1}\left[2-2\cos\frac{2\pi j}{bn}\right]$$
$$= \frac{1}{4bn}\prod_{\substack{j=1\\ j \bmod 4=1,3}}^{4bn-1}\left[4-2\cos\frac{2\pi j}{4bn}\right]\prod_{\substack{j=1\\ j \bmod 4=2}}^{4bn-1}\left[6-2\cos\frac{2\pi j}{4bn}\right]\times\prod_{j=1}^{bn-1}\left[2-2\cos\frac{2\pi j}{bn}\right]$$
$$= \frac{1}{4bn}\prod_{j=1}^{4bn-1}\left[4-2\cos\frac{2\pi j}{4bn}\right]\frac{1}{\prod\limits_{\substack{j=1\\ j \bmod 4=0,2}}^{4bn-1}\left[4-2\cos\frac{2\pi j}{4bn}\right]}\prod_{\substack{j=1\\ j \bmod 4=2}}^{4bn-1}\left[6-2\cos\frac{2\pi j}{4bn}\right]\prod_{j=1}^{bn-1}\left[2-2\cos\frac{2\pi j}{bn}\right]$$
$$= \frac{1}{4bn}\prod_{j=1}^{4bn-1}\left[4-2\cos\frac{2\pi j}{4bn}\right]\prod_{\substack{j=1\\ j \bmod 4=2}}^{4bn-1}\frac{6-2\cos\frac{2\pi j}{4bn}}{4-2\cos\frac{2\pi j}{4bn}}\prod_{j=1}^{bn-1}\frac{2-2\cos\frac{2\pi j}{bn}}{4-2\cos\frac{2\pi j}{bn}}$$
$$= \frac{1}{4bn}\prod_{j=1}^{4bn-1}\left[4-2\cos\frac{2\pi j}{4bn}\right]\prod_{\substack{j=1\\ j \bmod 2=0}}^{4bn-1}\frac{6-2\cos\frac{2\pi j}{4bn}}{4-2\cos\frac{2\pi j}{4bn}}\times\frac{1}{\prod\limits_{\substack{j=1\\ j \bmod 4=0}}^{4bn-1}\frac{6-2\cos\frac{2\pi j}{4bn}}{4-2\cos\frac{2\pi j}{4bn}}}\prod_{j=1}^{bn-1}\frac{2-2\cos\frac{2\pi j}{bn}}{4-2\cos\frac{2\pi j}{bn}}$$
$$= \frac{1}{4bn}\prod_{j=1}^{4bn-1}\left(4-2\cos\frac{2\pi j}{4bn}\right)\prod_{j=1}^{2bn-1}\frac{6-2\cos\frac{2\pi j}{2bn}}{4-2\cos\frac{2\pi j}{2bn}}\prod_{j=1}^{bn-1}\frac{2-2\cos\frac{2\pi j}{bn}}{6-2\cos\frac{2\pi j}{bn}}$$
$$= \frac{1}{4bn}U_{4bn-1}^2\left(\sqrt{\frac{4+2}{4}}\right)\frac{U_{2bn-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}{U_{2bn-1}^2\left(\sqrt{\frac{4+2}{4}}\right)}\frac{U_{bn-1}^2\left(\sqrt{\frac{2+2}{4}}\right)}{U_{bn-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}$$
$$= \frac{1}{4bn}U_{4bn-1}^2\left(\sqrt{\frac{3}{2}}\right)\frac{U_{2bn-1}^2\left(\sqrt{2}\right)}{U_{2bn-1}^2\left(\sqrt{\frac{3}{2}}\right)}\frac{U_{bn-1}^2(1)}{U_{bn-1}^2\left(\sqrt{2}\right)}$$

$$= \frac{bn}{4}\left[\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^{2bn}+\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)^{2bn}\right]^2\left[\left(\sqrt{2}+1\right)^{bn}+\left(\sqrt{2}-1\right)^{bn}\right]^2$$

$$= \frac{pn}{4^2}\left[\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^{\frac{p}{2}n}+\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)^{\frac{p}{2}n}\right]^2\left[\left(\sqrt{2}+1\right)^{\frac{p}{4}n}+\left(\sqrt{2}-1\right)^{\frac{p}{4}n}\right]^2,$$

and so

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{4}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{4}n}\right)} = \left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^{p}\left(\sqrt{2}+1\right)^{\frac{p}{2}}.$$

(d) If $\frac{p}{b}=6$, then repeating same operations as we did in (a) and making use of Formulas (4) and (3) yield

$$T\left(C_{pn}^{1,bn}\right) = \frac{1}{6bn}\prod_{j=1}^{6bn-1}\left[4-2\cos\frac{2\pi j}{6bn}-2\cos\frac{2\pi bnj}{6bn}\right]$$

$$= \frac{1}{6bn}\prod_{j=1}^{6bn-1}\left(3-2\cos\frac{2\pi j}{6bn}\right)\prod_{j=1}^{3bn-1}\frac{(5-2\cos\frac{2\pi j}{3bn})}{(3-2\cos\frac{2\pi j}{3bn})}\prod_{j=1}^{2bn-1}\frac{(6-2\cos\frac{2\pi j}{2bn})}{(3-2\cos\frac{2\pi j}{2bn})}$$

$$\prod_{j=1}^{bn-1}\frac{(2-2\cos\frac{2\pi j}{bn})(3-2\cos\frac{2\pi j}{bn})}{(6-2\cos\frac{2\pi j}{bn})(5-2\cos\frac{2\pi j}{bn})}$$

$$= \frac{1}{6bn}U_{6bn-1}^2\left(\sqrt{\frac{3+2}{4}}\right)\frac{U_{3bn-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}{U_{3bn-1}^2\left(\sqrt{\frac{3+2}{4}}\right)}\frac{U_{2bn-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}{U_{2bn-1}^2\left(\sqrt{\frac{3+2}{4}}\right)}\frac{U_{bn-1}^2\left(\sqrt{\frac{2+2}{4}}\right)}{U_{bn-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}\frac{U_{bn-1}^2\left(\sqrt{\frac{3+2}{4}}\right)}{U_{bn-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}$$

$$= \frac{1}{6bn}U_{6bn-1}^2\left(\sqrt{\frac{5}{4}}\right)\frac{U_{3bn-1}^2\left(\sqrt{\frac{7}{4}}\right)}{U_{3bn-1}^2\left(\sqrt{\frac{5}{4}}\right)}\frac{U_{2bn-1}^2\left(\sqrt{2}\right)}{U_{2bn-1}^2\left(\sqrt{\frac{5}{4}}\right)}\frac{U_{bn-1}^2(1)}{U_{bn-1}^2\left(\sqrt{2}\right)}\frac{U_{bn-1}^2\left(\sqrt{\frac{5}{4}}\right)}{U_{bn-1}^2\left(\sqrt{\frac{7}{4}}\right)}$$

$$= \frac{bn}{6}\left[\left(\sqrt{\frac{5}{4}}+\sqrt{\frac{1}{4}}\right)^{2bn}+\left(\sqrt{\frac{5}{4}}-\sqrt{\frac{1}{4}}\right)^{2bn}-1\right]^2\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2bn}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2bn}+1\right]^2$$
$$\times\left[\left(\sqrt{2}+1\right)^{bn}+\left(\sqrt{2}-1\right)^{bn}\right]^2$$

$$= \frac{pn}{6^2}\left[\left(\sqrt{\frac{5}{4}}+\sqrt{\frac{1}{4}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{5}{4}}-\sqrt{\frac{1}{4}}\right)^{\frac{p}{3}n}-1\right]^2\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n}+1\right]^2$$
$$\times\left[\left(\sqrt{2}+1\right)^{\frac{p}{6}n}+\left(\sqrt{2}-1\right)^{\frac{p}{6}n}\right]^2,$$

and so

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n}\right)} = \left(\sqrt{\frac{5}{4}}+\sqrt{\frac{1}{4}}\right)^{\frac{2p}{3}}\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{2p}{3}}\left(\sqrt{2}+1\right)^{\frac{p}{3}}.$$

Example 1. From [21] (Theorem 4) we see that

$$T\left(C_{2n}^{1,n}\right) = \frac{n}{2}\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2.$$

In our consideration this is the case that $p = 2, b = 1$ and from derivation (a)

$$\begin{aligned} T\left(C_{pn}^{1,bn}\right) &= T\left(C_{2n}^{1,n}\right) \\ &= \frac{pn}{2^2}\left[\left(\sqrt{2}+1\right)^{\frac{p}{2}n} + \left(\sqrt{2}-1\right)^{\frac{p}{2}n}\right]^2 \\ &= \frac{2n}{2^2}\left[\left(\sqrt{2}+1\right)^{\frac{2}{2}n} + \left(\sqrt{2}-1\right)^{\frac{2}{2}n}\right]^2 \\ &= \frac{n}{2}\left[\left(\sqrt{2}+1\right)^{n} + \left(\sqrt{2}-1\right)^{n}\right]^2. \end{aligned}$$

The results are identical. Comparing the asymptotic numbers of the formulas $T\left(C_{pn}^{1,bn}\right)$, one sees easily that in graphs $C_{pn}^{1,bn}$ satisfying $\frac{p}{b} \in \{2, 3, 4, 6\}$, the asymptotic value attains its maximum when $\frac{p}{b} = 6$.

Repeatedly applying the same technique, for different constraints of a, b, c, d, e, p, we can derive the following formulas (e) to (o). Because their derivations involve rather cumbersome transformations, we put them in Appendix.

(II) The case $T\left(C_{pn}^{1,bn,cn}\right)$, where b, c, p are any positive integers satisfying $\frac{p}{b}, \frac{p}{c} \in \{2, 3, 4, 6\}$.

(e) If $\frac{p}{b} = 6, \frac{p}{c} = 4$, then

$$\begin{aligned} T\left(C_{pn}^{1,bn,cn}\right) = \frac{pn}{12^2} &\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n} + \left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n} - 1\right]^2 \left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n} + \left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n} + 1\right]^2 \\ &\times \left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n} + \left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n} - 1\right]^2 \left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n} + \left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n}\right]^2 \\ &\times \left[\left(\sqrt{2}+1\right)^{\frac{p}{12}n} + \left(\sqrt{2}-1\right)^{\frac{p}{12}n}\right]^2, \end{aligned}$$

and so its asymptotics is

$$\lim_{n\to\infty} \frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{4}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{4}n}\right)} = \left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{p} \left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{3}} \left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}} \left(\sqrt{2}+1\right)^{\frac{p}{6}}.$$

(f) If $\frac{p}{b} = 6, \frac{p}{c} = 3$, then

$$T\left(C_{pn}^{1,bn,cn}\right) = \frac{pn}{6^2}\left[\left(\sqrt{2}+1\right)^{\frac{p}{2}n} + \left(\sqrt{2}-1\right)^{\frac{p}{2}n}\right]^2 \left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}n} + \left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}n} + 1\right]^2, \tag{5}$$

and so

$$\lim_{n\to\infty} \frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{3}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{3}n}\right)} = \left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{2p}{3}} \left(\sqrt{2}+1\right)^{p}.$$

(g) If $\frac{p}{b} = 6, \frac{p}{c} = 2$ then

$$\begin{aligned} T\left(C_{pn}^{1,bn,cn}\right) = \frac{pn}{6^2} &\left[\left(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}}\right)^{\frac{p}{3}n} + \left(\sqrt{\frac{9}{4}}-\sqrt{\frac{5}{4}}\right)^{\frac{p}{3}n} - 1\right]^2 \\ &\times \left[\left(\sqrt{3}+\sqrt{2}\right)^{\frac{p}{6}n} + \left(\sqrt{3}-\sqrt{2}\right)^{\frac{p}{6}n}\right]^2 \left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n} + \left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n} + 1\right]^2, \end{aligned}$$

and

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{2}n}\right)}=\left(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}}\right)^{\frac{2p}{3}}\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{2p}{3}}\left(\sqrt{3}+\sqrt{2}\right)^{\frac{p}{3}}.$$

(h) If $\frac{p}{b}=4$, $\frac{p}{c}=3$ then

$$\begin{aligned}T\left(C_{pn}^{1,bn,cn}\right)=\;&\frac{pn}{12^2}\left[\left(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{9}{4}}-\sqrt{\frac{5}{4}}\right)^{\frac{p}{3}n}-1\right]^2\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n}-1\right]^2\\&\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n}+1\right]^2\left[\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)^{\frac{p}{6}n}\right]^2\\&\times\left[\left(\sqrt{2}+1\right)^{\frac{p}{12}n}+\left(\sqrt{2}-1\right)^{\frac{p}{12}n}\right]^2,\end{aligned}$$

and so

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{4}(n+1),\frac{p}{3}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{4}n,\frac{p}{3}n}\right)}=\left(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}}\right)^{\frac{2p}{3}}\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{3}}\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^{\frac{p}{3}}\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}}\left(\sqrt{2}+1\right)^{\frac{p}{6}}.$$

(i) If $\frac{p}{b}=4$, $\frac{p}{c}=2$ then

$$T\left(C_{pn}^{1,bn,cn}\right)=\frac{pn}{4^2}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{2}n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{2}n}\right]^2\left[\left(\sqrt{2}+1\right)^{\frac{p}{4}n}+\left(\sqrt{2}-1\right)^{\frac{p}{4}n}\right]^2, \tag{6}$$

and so

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{4}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{4}n,\frac{p}{2}n}\right)}=\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{p}\left(\sqrt{2}+1\right)^{\frac{p}{2}}.$$

(j) If $\frac{p}{b}=3$, $\frac{p}{c}=2$ then

$$\begin{aligned}T\left(C_{pn}^{1,bn,cn}\right)=\;&\frac{pn}{6^2}\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{\frac{p}{3}n}-1\right]^2\\&\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n}+1\right]^2\left[\left(\sqrt{2}+1\right)^{\frac{p}{6}n}+\left(\sqrt{2}-1\right)^{\frac{p}{6}n}\right]^2,\end{aligned}$$

and

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{3}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{3}n,\frac{p}{2}n}\right)}=\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{2p}{3}}\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{2p}{3}}\left(\sqrt{2}+1\right)^{\frac{p}{3}}.$$

From the asymptotic numbers of $T\left(C_{pn}^{1,bn,cn}\right)$, we see easily that in the graphs $C_{pn}^{1,bn,cn}$ satisfying $\frac{p}{b},\frac{p}{c}\in\{2,3,4,6\}$ the asymptotic value attains its maximum when $\frac{p}{b}=6$, $\frac{p}{c}=4$.

(III) The case $T\left(C_{pn}^{1,bn,cn,dn}\right)$, where $\frac{p}{b},\frac{p}{c},\frac{p}{d}\in\{2,3,4,6\}$.

(k) If $\frac{p}{b}=6$, $\frac{p}{c}=4$, $\frac{p}{d}=3$ then

$$T\left(C_{pn}^{1,bn,cn,dn}\right)=\frac{pn}{12^2}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{2}n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{2}n}\right]^2\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n}+1\right]^2$$
$$\times\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{\frac{p}{6}n}-1\right]^2\left[\left(\sqrt{2}+1\right)^{\frac{p}{12}n}+\left(\sqrt{2}-1\right)^{\frac{p}{12}n}\right]^2,$$

and therefore

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{4}(n+1),\frac{p}{3}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{4}n,\frac{p}{3}n}\right)}=\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{4p}{3}}\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{p}{3}}\left(\sqrt{2}+1\right)^{\frac{p}{6}}.$$

(l) If $\frac{p}{b}=6$, $\frac{p}{c}=4$, $\frac{p}{d}=2$ then

$$T\left(C_{pn}^{1,bn,cn,dn}\right)=\frac{pn}{12^2}\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{2p}{3}n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{\frac{2p}{3}n}+1\right]^2$$
$$\times\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n}+1\right]^{-2}\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{\frac{p}{6}n}\right]^2$$
$$\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n}+1\right]^2\left[\left(\sqrt{2}+1\right)^{\frac{p}{12}n}+\left(\sqrt{2}-1\right)^{\frac{p}{12}n}\right]^2,$$

and

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{4}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{4}n,\frac{p}{2}n}\right)}=\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{p}\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{p}{3}}\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}}\left(\sqrt{2}+1\right)^{\frac{p}{6}}.$$

(m) If $\frac{p}{b}=6$, $\frac{p}{c}=3$, $\frac{p}{d}=2$

$$T\left(C_{pn}^{1,bn,cn,dn}\right)=\frac{pn}{6^2}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}n}+1\right]^2\left[\left(\sqrt{3}+\sqrt{2}\right)^{\frac{p}{2}n}+\left(\sqrt{3}-\sqrt{2}\right)^{\frac{p}{2}n}\right]^2,$$

implying that

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{3}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{3}n,\frac{p}{2}n}\right)}=\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{2p}{3}}(\sqrt{3}+\sqrt{2})^p.$$

(n) If $\frac{p}{b}=4$, $\frac{p}{c}=3$, $\frac{p}{d}=2$ then

$$T\left(C_{pn}^{1,bn,cn,dn}\right)=\frac{pn}{12^2}\left[\left(\sqrt{\frac{13}{4}}+\sqrt{\frac{9}{4}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{13}{4}}-\sqrt{\frac{9}{4}}\right)^{\frac{p}{3}n}-1\right]^2$$
$$\times\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n}-1\right]^2\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n}+1\right]^2$$
$$\times\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n}\right]^2\left[\left(\sqrt{2}+1\right)^{\frac{p}{12}n}+\left(\sqrt{2}-1\right)^{\frac{p}{12}n}\right]^2,$$

and

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{4}(n+1),\frac{p}{3}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{4}n,\frac{p}{3}n,\frac{p}{2}n}\right)} = \left(\sqrt{\frac{13}{4}}+\sqrt{\frac{9}{4}}\right)^{\frac{2p}{3}}\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{3}}\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}}\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}}\left(\sqrt{2}+1\right)^{\frac{p}{6}}.$$

From the asymptotics of $T\left(C_{pn}^{1,bn,cn,dn}\right)$, one sees that in graphs $C_{pn}^{1,bn,cn,dn}$ satisfying $\frac{p}{b},\frac{p}{c},\frac{p}{d}\in\{2,3,4,6\}$, the maximum value attains when $\frac{p}{b}=6$, $\frac{p}{c}=4$, $\frac{p}{d}=3$.

(IV) The case $T\left(C_{pn}^{1,bn,cn,dn,en}\right)$, where $\frac{p}{b},\frac{p}{c},\frac{p}{d},\frac{p}{e}\in\{2,3,4,6\}$.

(o) If $\frac{p}{b}=6$, $\frac{p}{c}=4$, $\frac{p}{d}=3$, $\frac{p}{e}=2$, then

$$\begin{aligned}T\left(C_{pn}^{1,bn,cn,dn,en}\right) &= \frac{pn}{12^2}\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{p}{2}n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{\frac{p}{2}n}\right]^2\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{\frac{p}{6}n}-1\right]^2\\ &\times\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n}+1\right]^2\left[\left(\sqrt{2}+1\right)^{\frac{p}{12}n}+\left(\sqrt{2}-1\right)^{\frac{p}{12}n}\right]^2,\end{aligned}$$

and

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{4}(n+1),\frac{p}{3}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{4}n,\frac{p}{3}n,\frac{p}{2}n}\right)} = \left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{4p}{3}}\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}}\left(\sqrt{2}+1\right)^{\frac{p}{6}}.$$

Observing the above discussions, we have the following theorem:

Corollary 1. *Let $0<a_1\le a_2\le\cdots\le a_4<p$ be integers and $\frac{p}{a_i}\in\{2,3,4,6\}$, $i=1,2,3,4$. Then*

$$\lim_{n\to\infty}T\left(C_{pn}^{1,a_1n,a_2n,\ldots,a_4n}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,a_1(n+1),a_2(n+1),\ldots,a_4(n+1)}\right)}{T\left(C_{pn}^{1,a_1n,a_2n,\ldots,a_4n}\right)}.$$

3. The simple formulas for the asymptotic values

Lemma 2. *Let $1\le b<p$ and $\frac{p}{b}$ be integers. Then*

$$\lim_{n\to\infty}T\left(C_{pn}^{1,bn}\right)^{\frac{1}{n}}=\prod_{t=0}^{\frac{p}{b}-1}\left(\sqrt{1+\sin^2\frac{\pi bt}{p}}+\sqrt{\sin^2\frac{\pi bt}{p}}\right)^{2b}.$$

Proof. From Theorem A (Theorem 3 in [10]), when $k=l=1$ and $s_1=1$, $a_1=b$, we have

$$\lim_{n\to\infty}T\left(C_{pn}^{1,bn}\right)^{\frac{1}{n}}=4^p\exp\left[\sum_{t=0}^{p-1}\int_0^1\ln\left(\sin^2\pi x+\sin^2\frac{\pi bt}{p}\right)dx\right].$$

Set $\frac{p}{b}=k$. Making use of the relation $\sin^2(n\pi+\alpha)=\sin^2\alpha$ where n is an integer, we have

$$\begin{aligned}\lim_{n\to\infty}T\left(C_{pn}^{1,bn}\right)^{\frac{1}{n}} &= 4^p\exp\left[\sum_{t=0}^{p-1}\int_0^1\ln\left(\sin^2\pi x+\sin^2\frac{\pi t}{k}\right)dx\right]\qquad(\text{letting } t=mk+t')\\ &= 4^p\exp\left[\sum_{m=0}^{\frac{p}{k}-1}\sum_{t'=0}^{k-1}\int_0^1\ln\left(\sin^2\pi x+\sin^2\frac{(mk+t')}{k}\pi\right)dx\right]\end{aligned}$$

$$= 4^p \exp\left[\sum_{m=0}^{\frac{p}{k}-1}\sum_{t'=0}^{k-1}\int_0^1 \ln\left(\sin^2 \pi x + \sin^2\left(m\pi + \frac{t'}{k}\pi\right)\right)dx\right]$$
$$= 4^p \exp\left[\sum_{m=0}^{\frac{p}{k}-1}\sum_{t'=0}^{k-1}\int_0^1 \ln\left(\sin^2 \pi x + \sin^2 \frac{t'}{k}\pi\right)dx\right]$$
$$= 4^p \exp\left[\frac{p}{k}\sum_{t'=0}^{k-1}\int_0^1 \ln\left(\sin^2 \pi x + \sin^2 \frac{t'}{k}\pi\right)dx\right].$$

We first evaluate $\int_0^1 \ln(\sin^2 \pi x + \sin^2 \frac{t'\pi}{k})dx$, where $t' = 0, 1, 2, \ldots, k-1$. Using the relation $\sin^2 \pi x + \sin^2 \frac{t'\pi}{k} = 1+\sin^2 \frac{t'\pi}{k} - \cos^2 \pi x$, and the formulas for the Chebyshev polynomial of the second kind addressed in Introduction, we have

$$\int_0^1 \ln\left(\sin^2 \pi x + \sin^2 \frac{t'\pi}{k}\right)dx = \int_0^1 \ln\left(1+\sin^2 \frac{t'\pi}{k} - \cos^2 \pi x\right)dx$$
$$= \lim_{n\to\infty}\left(\sum_{j=1}^{n-1}\ln\left(1+\sin^2\frac{t'\pi}{k} - \cos^2\frac{\pi j}{n}\right)\right)\times\frac{1}{n}$$
$$= \lim_{n\to\infty}\ln\left(\prod_{j=1}^{n-1}\left(1+\sin^2\frac{t'\pi}{k} - \cos^2\frac{\pi j}{n}\right)\right)\times\frac{1}{n}$$
$$= \lim_{n\to\infty}\ln\left(\frac{U_{n-1}^2\left(\sqrt{1+\sin^2\frac{t'\pi}{k}}\right)}{4^{n-1}}\right)\times\frac{1}{n}$$
$$= \ln\frac{\left(\sqrt{1+\sin^2\frac{t'\pi}{k}}+\sqrt{\sin^2\frac{t'\pi}{k}}\right)^2}{4}.$$

Consequently,

$$\lim_{n\to\infty} T\left(C_{pn}^{1,bn}\right)^{\frac{1}{n}} = 4^p \exp\left[\frac{p}{k}\sum_{t'=0}^{k-1}\int_0^1 \ln\left(\sin^2 \pi x + \sin^2 \frac{t'}{k}\pi\right)dx\right]$$
$$= 4^p \exp\left[\frac{p}{k}\sum_{t'=0}^{k-1}\ln\frac{\left(\sqrt{1+\sin^2\frac{t'\pi}{k}}+\sqrt{\sin^2\frac{t'\pi}{k}}\right)^2}{4}\right]$$
$$= 4^p \exp\left[\ln\prod_{t'=0}^{k-1}\frac{\left(\sqrt{1+\sin^2\frac{t'\pi}{k}}+\sqrt{\sin^2\frac{t'\pi}{k}}\right)^{\frac{2p}{k}}}{4^{\frac{p}{k}}}\right]$$
$$= 4^p\cdot\frac{1}{4^{\frac{p}{k}\cdot k}}\prod_{t'=0}^{k-1}\left(\sqrt{1+\sin^2\frac{t'\pi}{k}}+\sqrt{\sin^2\frac{t'\pi}{k}}\right)^{\frac{2p}{k}}$$
$$= \prod_{t'=0}^{\frac{p}{b}-1}\left(\sqrt{1+\sin^2\frac{\pi b t'}{p}}+\sqrt{\sin^2\frac{\pi b t'}{p}}\right)^{2b}.$$

The proof is completed.

Lemma 3. *Let $0 < b \le c < p$, $\frac{p}{b}$ and $\frac{p}{c}$ be integers. Then*

$$\lim_{n\to\infty} T\left(C_{pn}^{1,bn,cn}\right)^{\frac{1}{n}} = \prod_{t=0}^{k-1}\left(\sqrt{1+\sin^2\frac{\pi bt}{p}+\sin^2\frac{\pi ct}{p}}+\sqrt{\sin^2\frac{\pi bt}{p}+\sin^2\frac{\pi ct}{p}}\right)^{\frac{2p}{k}}$$

in which $k = lcm(\frac{p}{b}, \frac{p}{c})$, where lcm denotes the least common multiple.

Proof. Set $\frac{p}{b} = k_1$, $\frac{p}{c} = k_2$. Then $\frac{k}{k_1}$, $\frac{k}{k_2}$ and $\frac{p}{k}$ are integers. Making use of Theorem A and the properties of Chebyshev polynomial of the second kind, similar to the proof of Lemma 2, we have

$$\begin{aligned}
\lim_{n\to\infty} T\left(C_{pn}^{1,bn,cn}\right)^{\frac{1}{n}} &= 4^p \exp\left[\sum_{t=0}^{p-1}\int_0^1 \ln\left(\sin^2 \pi x + \sin^2 \frac{\pi bt}{p} + \sin^2 \frac{\pi ct}{p}\right) dx\right] \\
&= 4^p \exp\left[\sum_{t=0}^{p-1}\int_0^1 \ln\left(\sin^2 \pi x + \sin^2 \frac{\pi t}{k_1} + \sin^2 \frac{\pi t}{k_2}\right) dx\right] \\
&= 4^p \exp\left[\sum_{m=0}^{\frac{p}{k}-1}\sum_{t'=0}^{k-1}\int_0^1 \ln\left(\sin^2 \pi x + \sin^2 \frac{(mk+t')}{k_1}\pi + \sin^2 \frac{(mk+t')}{k_2}\pi\right) dx\right] \\
&= 4^p \exp\left[\sum_{m=0}^{\frac{p}{k}-1}\sum_{t'=0}^{k-1}\int_0^1 \ln\left(\sin^2 \pi x + \sin^2 \left(\frac{mk}{k_1}\pi + \frac{t'}{k_1}\pi\right) + \sin^2 \left(\frac{mk}{k_2}\pi + \frac{t'}{k_2}\pi\right)\right) dx\right] \\
&= 4^p \exp\left[\sum_{m=0}^{\frac{p}{k}-1}\sum_{t'=0}^{k-1}\int_0^1 \ln\left(\sin^2 \pi x + \sin^2 \frac{t'}{k_1}\pi + \sin^2 \frac{t'}{k_2}\pi\right) dx\right] \\
&= 4^p \exp\left[\frac{p}{k}\sum_{t'=0}^{k-1}\int_0^1 \ln\left(\sin^2 \pi x + \sin^2 \frac{t'}{k_1}\pi + \sin^2 \frac{t'}{k_2}\pi\right) dx\right] \\
&= 4^p \exp\left[\frac{p}{k}\sum_{t'=0}^{k-1}\ln \frac{\left(\sqrt{1+\sin^2 \frac{t'\pi}{k_1} + \sin^2 \frac{t'\pi}{k_2}} + \sqrt{\sin^2 \frac{t'\pi}{k_1} + \sin^2 \frac{t'\pi}{k_2}}\right)^2}{4}\right] \\
&= 4^p \exp\left[\ln \prod_{t'=0}^{k-1} \frac{\left(\sqrt{1+\sin^2 \frac{t'\pi}{k_1} + \sin^2 \frac{t'\pi}{k_2}} + \sqrt{\sin^2 \frac{t'\pi}{k_1} + \sin^2 \frac{t'\pi}{k_2}}\right)^{\frac{2p}{k}}}{4^{\frac{p}{k}}}\right] \\
&= 4^p \cdot \frac{1}{4^{\frac{p}{k}\cdot k}} \prod_{t'=0}^{k-1}\left(\sqrt{1+\sin^2 \frac{t'\pi}{k_1} + \sin^2 \frac{t'\pi}{k_2}} + \sqrt{\sin^2 \frac{t'\pi}{k_1} + \sin^2 \frac{t'\pi}{k_2}}\right)^{\frac{2p}{k}} \\
&= \prod_{t'=0}^{k-1}\left(\sqrt{1+\sin^2 \frac{\pi bt'}{p} + \sin^2 \frac{\pi ct'}{p}} + \sqrt{\sin^2 \frac{\pi bt'}{p} + \sin^2 \frac{\pi ct'}{p}}\right)^{\frac{2p}{k}}.
\end{aligned}$$

The lemma is proven. Continuing the same argument, for any integer $l > 2$ we achieve at

Theorem 4. *Let $0 < a_1 \le \cdots \le a_l < p$ and $\frac{p}{a_i}$ be integers $i = 1, 2, \ldots, l$. Then*

$$\lim_{n\to\infty} T\left(C_{pn}^{1,a_1n,a_2n,\ldots,a_ln}\right)^{\frac{1}{n}} = \prod_{t=0}^{k-1}\left(\sqrt{1+\sum_{i=1}^{l}\sin^2 \frac{\pi a_i t}{p}} + \sqrt{\sum_{i=1}^{l}\sin^2 \frac{\pi a_i t}{p}}\right)^{\frac{2p}{k}},$$

where $k = lcm(\frac{p}{a_1}, \frac{p}{a_2}, \ldots, \frac{p}{a_l})$, in which lcm denotes the least common multiple.

The following theorem reveals the monotonic property of the asymptotic value of the number of spanning trees.

Theorem 5. *Let $1 \le b < p$ and $\frac{p}{b}$ be integers. If b is fixed, then the asymptotics*

$$\lim_{n\to\infty} T\left(C_{pn}^{1,bn}\right)^{\frac{1}{n}} = \prod_{t=0}^{\frac{p}{b}-1}\left(\sqrt{1+\sin^2 \frac{\pi bt}{p}} + \sqrt{\sin^2 \frac{\pi bt}{p}}\right)^{2b}$$

is an increasing function of p.

Proof. Set $k = \frac{p}{b}$.

(1) If k is even, making use of the basic formulas $\sin(\pi-\alpha)=\sin\alpha$ and $\sin(\frac{\pi}{2}-\alpha)=\cos\alpha$, we have

$$\begin{aligned}
\lim_{n\to\infty} T\left(C_{pn}^{1,bn}\right)^{\frac{1}{n}} &= \prod_{t=0}^{\frac{p}{b}-1}\left(\sqrt{1+\sin^2\frac{\pi bt}{p}}+\sqrt{\sin^2\frac{\pi bt}{p}}\right)^{2b} \\
&= \prod_{t=1}^{k-1}\left(\sqrt{1+\sin^2\frac{t\pi}{k}}+\sqrt{\sin^2\frac{t\pi}{k}}\right)^{2b} \\
&= \prod_{t=1}^{\frac{k-2}{2}}\left(\sqrt{1+\sin^2\frac{t\pi}{k}}+\sqrt{\sin^2\frac{t\pi}{k}}\right)^{2b}\left(\sqrt{1+\sin^2\frac{k\pi}{2k}}+\sqrt{\sin^2\frac{k\pi}{2k}}\right)^{2b} \\
&\quad\times\prod_{t=\frac{k+2}{2}}^{k-1}\left(\sqrt{1+\sin^2\frac{t\pi}{k}}+\sqrt{\sin^2\frac{t\pi}{k}}\right)^{2b} \\
&= \prod_{t=1}^{\frac{k-2}{2}}\left(\sqrt{1+\sin^2\frac{t\pi}{k}}+\sqrt{\sin^2\frac{t\pi}{k}}\right)^{2b}\left(\sqrt{2}+1\right)^{2b} \\
&\quad\times\prod_{t'=1}^{\frac{k-2}{2}}\left(\sqrt{1+\sin^2\left(\pi-\frac{t'\pi}{k}\right)}+\sqrt{\sin^2\left(\pi-\frac{t'\pi}{k}\right)}\right)^{2b} \\
&= \prod_{t=1}^{\frac{k-2}{2}}\left(\sqrt{1+\sin^2\frac{t\pi}{k}}+\sqrt{\sin^2\frac{t\pi}{k}}\right)^{2b}\left(\sqrt{2}+1\right)^{2b}\prod_{t'=1}^{\frac{k-2}{2}}\left(\sqrt{1+\sin^2\frac{t'\pi}{k}}+\sqrt{\sin^2\frac{t'\pi}{k}}\right)^{2b} \\
&= \prod_{t=1}^{\frac{k-2}{2}}\left(\sqrt{1+\sin^2\frac{t\pi}{k}}+\sqrt{\sin^2\frac{t\pi}{k}}\right)^{4b}\left(\sqrt{2}+1\right)^{2b}\quad\left(\text{letting } t=\frac{k}{2}-j\right) \\
&= \prod_{j=1}^{\frac{k-2}{2}}\left(\sqrt{1+\sin^2\left(\frac{\pi}{2}-\frac{j\pi}{k}\right)}+\sqrt{\sin^2\left(\frac{\pi}{2}-\frac{j\pi}{k}\right)}\right)^{4b}\left(\sqrt{2}+1\right)^{2b} \\
&= \prod_{j=1}^{\frac{k-2}{2}}\left(\sqrt{1+\cos^2\frac{j\pi}{k}}+\sqrt{\cos^2\frac{j\pi}{k}}\right)^{4b}\left(\sqrt{2}+1\right)^{2b} \\
&= \prod_{j=1}^{\frac{\frac{p}{b}-2}{2}}\left(\sqrt{1+\cos^2\frac{jb\pi}{p}}+\sqrt{\cos^2\frac{jb\pi}{p}}\right)^{4b}\left(\sqrt{2}+1\right)^{2b}.
\end{aligned}$$

Since b is fixed, for each fixed $j\in\{1,2,\ldots.\}$ and $\cos\frac{jb\pi}{p}$ is an increasing function of p, we conclude that

$$\lim_{n\to\infty} T\left(C_{pn}^{1,bn}\right)^{\frac{1}{n}} = \prod_{t=0}^{\frac{p}{b}-1}\left(\sqrt{1+\sin^2\frac{\pi bt}{p}}+\sqrt{\sin^2\frac{\pi bt}{p}}\right)^{2b}$$

is an increasing function of p.

(2) if k is odd,

$$\begin{aligned}
\lim_{n\to\infty} T\left(C_{pn}^{1,bn}\right)^{\frac{1}{n}} &= \prod_{t=0}^{\frac{p}{b}-1}\left(\sqrt{1+\sin^2\frac{\pi bt}{p}}+\sqrt{\sin^2\frac{\pi bt}{p}}\right)^{2b} \\
&= \prod_{t=1}^{k-1}\left(\sqrt{1+\sin^2\frac{t\pi}{k}}+\sqrt{\sin^2\frac{t\pi}{k}}\right)^{2b} \\
&= \prod_{t=1}^{\frac{k-1}{2}}\left(\sqrt{1+\sin^2\frac{t\pi}{k}}+\sqrt{\sin^2\frac{t\pi}{k}}\right)^{2b}\prod_{t=\frac{k+1}{2}}^{k-1}\left(\sqrt{1+\sin^2\frac{t\pi}{k}}+\sqrt{\sin^2\frac{t\pi}{k}}\right)^{2b}
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{t=1}^{\frac{k-1}{2}} \left(\sqrt{1+\sin^2\frac{t\pi}{k}}+\sqrt{\sin^2\frac{t\pi}{k}}\right)^{2b} \prod_{t'=1}^{\frac{k-1}{2}} \left(\sqrt{1+\sin^2\left(\pi-\frac{t'\pi}{k}\right)}+\sqrt{\sin^2\left(\pi-\frac{t'\pi}{k}\right)}\right)^{2b}\\
&= \prod_{t=1}^{\frac{k-1}{2}} \left(\sqrt{1+\sin^2\frac{t\pi}{k}}+\sqrt{\sin^2\frac{t\pi}{k}}\right)^{2b} \prod_{t'=1}^{\frac{k-1}{2}} \left(\sqrt{1+\sin^2\frac{t'\pi}{k}}+\sqrt{\sin^2\frac{t'\pi}{k}}\right)^{2b}\\
&= \prod_{t=1}^{\frac{k-1}{2}} \left(\sqrt{1+\sin^2\frac{t\pi}{k}}+\sqrt{\sin^2\frac{t\pi}{k}}\right)^{4b}\\
&= \prod_{j=1}^{\frac{k-1}{2}} \left(\sqrt{1+\sin^2\left(\frac{\pi}{2}-\frac{(j-\frac{1}{2})\pi}{k}\right)}+\sqrt{\sin^2\left(\frac{\pi}{2}-\frac{(j-\frac{1}{2})\pi}{k}\right)}\right)^{4b}\\
&= \prod_{j=1}^{\frac{k-1}{2}} \left(\sqrt{1+\cos^2\frac{(j-\frac{1}{2})\pi}{k}}+\sqrt{\cos^2\frac{(j-\frac{1}{2})\pi}{k}}\right)^{4b}\\
&= \prod_{j=1}^{\frac{\frac{p}{b}-1}{2}} \left(\sqrt{1+\cos^2\frac{(j-\frac{1}{2})b\pi}{p}}+\sqrt{\cos^2\frac{(j-\frac{1}{2})b\pi}{p}}\right)^{4b}.
\end{aligned}
$$

Similarly, since $\frac{(j-\frac{1}{2})\pi}{k} \in (0, \frac{\pi}{2})$, $\cos\frac{(j-\frac{1}{2})\pi}{k}$ is an increasing function of k, and so, if b is given and $\frac{p}{b}$ is odd,

$$
\lim_{n\to\infty} T\left(C_{pn}^{1,bn}\right)^{\frac{1}{n}} = \prod_{t=0}^{\frac{p}{b}-1} \left(\sqrt{1+\sin^2\frac{\pi bt}{p}}+\sqrt{\sin^2\frac{\pi bt}{p}}\right)^{2b}
$$

is an increasing function of p. The proof is therefore complete. From the above discussion one sees that the minimum value of $\lim_{n\to\infty} T\left(C_{pn}^{1,bn}\right)^{\frac{1}{n}}$, as a function of p, b, achieves when $\frac{p}{b}=2$. That is,

$$
\begin{aligned}
\lim_{n\to\infty} T\left(C_{pn}^{1,bn}\right)^{\frac{1}{n}}_{min} &= \prod_{t=0}^{2-1} \left(\sqrt{1+\sin^2\frac{t\pi}{2}}+\sqrt{\sin^2\frac{t\pi}{2}}\right)^{2b}\\
&= \left(\sqrt{1+\sin^2\frac{\pi}{2}}+\sqrt{\sin^2\frac{\pi}{2}}\right)^{2b}\\
&= \left(\sqrt{2}+1\right)^{2b}.
\end{aligned}
$$

4. Concluding remarks

In this paper we derived simple and explicit formulas for the number of spanning trees in circulant graphs of the form $C_{pn}^{1,a_1n,a_2n,\ldots,a_ln}$. We then obtained simple asymptotics for the number of spanning trees in such circulant graphs. Extending the technique can obtain the simplified formulas for the limiting asymptotics of the number of spanning trees for more general circulant graphs.

We conclude with a question about the growth rate of the number of spanning trees. From Theorem 4 we have

$$
\lim_{n\to\infty} T\left(C_{pn}^{1,a_1n,a_2n,\ldots,a_ln}\right)^{\frac{1}{n}} = \prod_{t=0}^{k-1} \left(\sqrt{1+\sum_{i=1}^{l}\sin^2\frac{\pi a_i t}{p}}+\sqrt{\sum_{i=1}^{l}\sin^2\frac{\pi a_i t}{p}}\right)^{\frac{2p}{k}},
$$

where $k = \operatorname{lcm}(\frac{p}{a_1}, \frac{p}{a_2}, \ldots, \frac{p}{a_l})$, $(i = 1, 2, \ldots, l)$, in which lcm denotes the least common multiple. Note that the limit represents the average growth rate of the number of spanning trees of the circulants. *Is it possible to find the values or relations of the jumps that maximize or minimize the average growth rate of the number of spanning trees among all families of $2k$-regular circulant graphs?* To the best of our knowledge, this problem has only been addressed [13] for directed circulant graphs with $k = 2$ (and with only partial solutions so far). It would be interesting to try and solve this more generally.

Acknowledgments

The authors would like to thank the referees for carefully reviewing on our original manuscript and their several valuable suggestions towards improving the presentation of the paper. The fourth author would also like to thank Professor Robert Acar for his discussions and comments on some of the results obtained.

Appendix

Below we derive the formulas for (e) to (o), addressed in Section 2.

(II) The case $T\left(C_{pn}^{1,bn,cn}\right)$, where b, c, p are any positive integers satisfying $\frac{p}{b}, \frac{p}{c} \in \{2, 3, 4, 6\}$.

(e) If $\frac{p}{b} = 6$, $\frac{p}{c} = 4$, then from the first equation in Lemma 1 and by utilizing Formulas (4) and (3) we have

$$
\begin{aligned}
T\left(C_{pn}^{1,bn,cn}\right) &= \frac{1}{pn}\prod_{j=1}^{pn-1}\left[6-2\cos\frac{2\pi j}{pn}-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}\right] \\
&= \frac{1}{pn}\prod_{j=1}^{pn-1}\left(5-2\cos\frac{2\pi j}{pn}\right)\prod_{j=1}^{\frac{p}{2}n-1}\frac{\left(9-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}{\left(5-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}\prod_{j=1}^{\frac{p}{3}n-1}\frac{\left(8-2\cos\frac{2\pi j}{\frac{p}{3}n}\right)}{\left(5-2\cos\frac{2\pi j}{\frac{p}{3}n}\right)} \\
&\times\prod_{j=1}^{\frac{p}{4}n-1}\frac{\left(5-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)}{\left(9-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)}\prod_{j=1}^{\frac{p}{6}n-1}\frac{\left(5-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)\left(6-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}{\left(8-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)\left(9-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)} \\
&\times\prod_{j=1}^{\frac{p}{12}n-1}\frac{\left(2-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)\left(9-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)}{\left(6-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)\left(5-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)} \\
&= \frac{1}{pn}U_{pn-1}^2\left(\sqrt{\frac{5+2}{4}}\right)\frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}\frac{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{8+2}{4}}\right)}{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}\frac{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)} \\
&\times\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{8+2}{4}}\right)}\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}\frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{2+2}{4}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}\frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)} \\
&= \frac{1}{pn}U_{pn-1}^2\left(\sqrt{\frac{7}{4}}\right)\frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{11}{4}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{7}{4}}\right)}\frac{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{5}{2}}\right)}{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{7}{4}}\right)}\frac{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{7}{4}}\right)}{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{11}{4}}\right)} \\
&\times\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{7}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{5}{2}}\right)}\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{2}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{11}{4}}\right)}\frac{U_{\frac{p}{12}n-1}^2(\sqrt{1})}{U_{\frac{p}{12}n-1}^2\left(\sqrt{2}\right)}\frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{11}{4}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{7}{4}}\right)} \\
&= \frac{pn}{12^2}\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n}-1\right]^2\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n}+1\right]^2 \\
&\times\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n}-1\right]^2\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n}\right]^2 \\
&\times\left[\left(\sqrt{2}+1\right)^{\frac{p}{12}n}+\left(\sqrt{2}-1\right)^{\frac{p}{12}n}\right]^2,
\end{aligned}
$$

and so its asymptotics is

$$
\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{4}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{4}n}\right)} = \left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{p}\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{3}}\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}}\left(\sqrt{2}+1\right)^{\frac{p}{6}}.
$$

(f) If $\frac{p}{b}=6$, $\frac{p}{c}=3$, then

$$\begin{aligned}
T\left(C_{pn}^{1,bn,cn}\right) &= \frac{1}{pn}\prod_{j=1}^{pn-1}\left[6-2\cos\frac{2\pi j}{pn}-2\cos\frac{\pi j}{3}-2\cos\frac{2\pi j}{3}\right]\\
&= \frac{1}{pn}\prod_{j=1}^{pn-1}\left(6-2\cos\frac{2\pi j}{pn}\right)\prod_{j=1}^{\frac{p}{6}n-1}\frac{\left(2-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}{\left(8-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}\prod_{j=1}^{\frac{p}{2}n-1}\frac{\left(8-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}{\left(6-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}\\
&= \frac{1}{pn}U_{pn-1}^2\left(\sqrt{\frac{6+2}{4}}\right)\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{2+2}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{8+2}{4}}\right)}\frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{8+2}{4}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}\\
&= \frac{1}{pn}U_{pn-1}^2\left(\sqrt{2}\right)\frac{U_{\frac{p}{6}n-1}^2(1)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{5}{2}}\right)}\frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{5}{2}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{2}\right)}\\
&= \frac{pn}{6^2}\left[\left(\sqrt{2}+1\right)^{\frac{p}{2}n}+\left(\sqrt{2}-1\right)^{\frac{p}{2}n}\right]^2\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}n}+1\right]^2,
\end{aligned}$$

and so

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{3}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{3}n}\right)}=\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{2p}{3}}\left(\sqrt{2}+1\right)^{p}.$$

(g) If $\frac{p}{b}=6$, $\frac{p}{c}=2$, then, similarly, we have that

$$\begin{aligned}
T\left(C_{pn}^{1,bn,cn}\right) &= \frac{1}{pn}\prod_{j=1}^{pn-1}\left[6-2\cos\frac{2\pi j}{pn}-2\cos\frac{\pi j}{3}-2\cos\pi j\right]\\
&= \frac{1}{pn}\prod_{j=1}^{pn-1}\left(7-2\cos\frac{2\pi j}{pn}\right)\prod_{j=1}^{\frac{p}{6}n-1}\frac{\left(7-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)\left(2-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}{\left(10-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)\left(5-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}\\
&\quad\times\prod_{j=1}^{\frac{p}{3}n-1}\frac{\left(10-2\cos\frac{2\pi j}{\frac{p}{3}n}\right)}{\left(7-2\cos\frac{2\pi j}{\frac{p}{3}n}\right)}\prod_{j=1}^{\frac{p}{2}n-1}\frac{\left(5-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}{\left(7-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}\\
&= \frac{1}{pn}U_{pn-1}^2\left(\sqrt{\frac{7+2}{4}}\right)\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{7+2}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{10+2}{4}}\right)}\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{2+2}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}\frac{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{10+2}{4}}\right)}{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{7+2}{4}}\right)}\frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{7+2}{4}}\right)}\\
&= \frac{1}{pn}U_{pn-1}^2\left(\sqrt{\frac{9}{4}}\right)\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{9}{4}}\right)}{U_{\frac{p}{6}n-1}^2(\sqrt{3})}\frac{U_{\frac{p}{6}n-1}^2(1)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{7}{4}}\right)}\frac{U_{\frac{p}{3}n-1}^2(\sqrt{3})}{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{9}{4}}\right)}\frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{7}{4}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{9}{4}}\right)}\\
&= \frac{pn}{6^2}\left[\left(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{9}{4}}-\sqrt{\frac{5}{4}}\right)^{\frac{p}{3}n}-1\right]^2\\
&\quad\times\left[\left(\sqrt{3}+\sqrt{2}\right)^{\frac{p}{6}n}+\left(\sqrt{3}-\sqrt{2}\right)^{\frac{p}{6}n}\right]^2\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n}+1\right]^2,
\end{aligned}$$

and

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{2}n}\right)}=\left(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}}\right)^{\frac{2p}{3}}\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{2p}{3}}(\sqrt{3}+\sqrt{2})^{\frac{p}{3}}.$$

(h) If $\frac{p}{b}=4,\ \frac{p}{c}=3$ then

$$\begin{aligned}
T\left(C_{pn}^{1,bn,cn}\right) &= \frac{1}{pn}\prod_{j=1}^{pn-1}\left[6-2\cos\frac{2\pi j}{pn}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}\right]\\
&= \frac{1}{pn}\prod_{j=1}^{pn-1}\left(7-2\cos\frac{2\pi j}{pn}\right)\prod_{j=1}^{\frac{p}{2}n-1}\frac{\left(9-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}{\left(7-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}\prod_{j=1}^{\frac{p}{4}n-1}\frac{\left(5-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)}{\left(9-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)}\prod_{j=1}^{\frac{p}{3}n-1}\frac{\left(4-2\cos\frac{2\pi j}{\frac{p}{3}n}\right)}{\left(7-2\cos\frac{2\pi j}{\frac{p}{3}n}\right)}\\
&\quad\times\prod_{j=1}^{\frac{p}{6}n-1}\frac{\left(6-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)\left(7-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}{\left(9-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)\left(4-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}\prod_{j=1}^{\frac{p}{12}n-1}\frac{\left(2-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)\left(9-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)}{\left(5-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)\left(6-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)}\\
&= \frac{1}{pn}U_{pn-1}^2\left(\sqrt{\frac{7+2}{4}}\right)\frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{7+2}{4}}\right)}\frac{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}\frac{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{4+2}{4}}\right)}{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{7+2}{4}}\right)}\\
&\quad\times\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{7+2}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{4+2}{4}}\right)}\frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{2+2}{4}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}\frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}\\
&= \frac{1}{pn}U_{pn-1}^2\left(\sqrt{\frac{9}{4}}\right)\frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{11}{4}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{9}{4}}\right)}\frac{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{7}{4}}\right)}{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{11}{4}}\right)}\frac{U_{\frac{p}{3}n-1}^2(\sqrt{\frac{3}{2}})}{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{9}{4}}\right)}\\
&\quad\times\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{2}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{3}{2}}\right)}\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{9}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{11}{4}}\right)}\frac{U_{\frac{p}{12}n-1}^2(1)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{7}{4}}\right)}\frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{11}{4}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{2}\right)}\\
&= \frac{pn}{12^2}\left[\left(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{9}{4}}-\sqrt{\frac{5}{4}}\right)^{\frac{p}{3}n}-1\right]^2\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n}-1\right]^2\\
&\quad\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n}+1\right]^2\left[\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)^{\frac{p}{6}n}\right]^2\\
&\quad\times\left[\left(\sqrt{2}+1\right)^{\frac{p}{12}n}+\left(\sqrt{2}-1\right)^{\frac{p}{12}n}\right]^2,
\end{aligned}$$

and so

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{4}(n+1),\frac{p}{3}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{4}n,\frac{p}{3}n}\right)}=\left(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}}\right)^{\frac{2p}{3}}\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{3}}\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^{\frac{p}{3}}\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}}\left(\sqrt{2}+1\right)^{\frac{p}{6}}.$$

(i) If $\frac{p}{b}=4,\ \frac{p}{c}=2$, then

$$\begin{aligned}
T\left(C_{pn}^{1,bn,cn}\right) &= \frac{1}{pn}\prod_{j=1}^{pn-1}\left[6-2\cos\frac{2\pi j}{pn}-2\cos\frac{\pi j}{2}-2\cos\pi j\right]\\
&= \frac{1}{pn}\prod_{j=1}^{pn-1}\left(8-2\cos\frac{2\pi j}{pn}\right)\prod_{j=1}^{\frac{p}{2}n-1}\frac{\left(6-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}{\left(8-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}\prod_{j=1}^{\frac{p}{4}n-1}\frac{\left(2-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)}{\left(6-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)}
\end{aligned}$$

$$= \frac{1}{pn} U_{pn-1}^2\left(\sqrt{\frac{8+2}{4}}\right) \frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{8+2}{4}}\right)} \frac{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{2+2}{4}}\right)}{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}$$

$$= \frac{1}{pn} U_{pn-1}^2\left(\sqrt{\frac{5}{2}}\right) \frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{2}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{5}{2}}\right)} \frac{U_{\frac{p}{4}n-1}^2(1)}{U_{\frac{p}{4}n-1}^2\left(\sqrt{2}\right)}$$

$$= \frac{pn}{4^2}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{2}n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{2}n}\right]^2 \left[\left(\sqrt{2}+1\right)^{\frac{p}{4}n}+\left(\sqrt{2}-1\right)^{\frac{p}{4}n}\right]^2,$$

and so

$$\lim_{n\to\infty} \frac{T\left(C_{p(n+1)}^{1,\frac{p}{4}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{4}n,\frac{p}{2}n}\right)} = \left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{p}\left(\sqrt{2}+1\right)^{\frac{p}{2}}.$$

(j) If $\frac{p}{b}=3$, $\frac{p}{c}=2$, then

$$T\left(C_{pn}^{1,bn,cn}\right) = \frac{1}{pn}\prod_{j=1}^{pn-1}\left[6-2\cos\frac{2\pi j}{pn}-2\cos\frac{2\pi j}{3}-2\cos\pi j\right]$$

$$= \frac{1}{pn}\prod_{j=1}^{pn-1}\left(9-2\cos\frac{2\pi j}{pn}\right)\prod_{j=1}^{\frac{p}{3}n-1}\frac{\left(6-2\cos\frac{2\pi j}{\frac{p}{3}n}\right)}{\left(9-2\cos\frac{2\pi j}{\frac{p}{3}n}\right)}$$

$$\times\prod_{j=1}^{\frac{p}{2}n-1}\frac{\left(5-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}{\left(9-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}\prod_{j=1}^{\frac{p}{6}n-1}\frac{\left(2-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)\left(9-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}{\left(5-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)\left(6-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}$$

$$= \frac{1}{pn}U_{pn-1}^2\left(\sqrt{\frac{9+2}{4}}\right)\frac{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}\frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{2+2}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}$$

$$= \frac{1}{pn}U_{pn-1}^2\left(\sqrt{\frac{11}{4}}\right)\frac{U_{\frac{p}{3}n-1}^2\left(\sqrt{2}\right)}{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{11}{4}}\right)}\frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{7}{4}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{11}{4}}\right)}\frac{U_{\frac{p}{6}n-1}^2(1)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{7}{4}}\right)}\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{11}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{2}\right)}$$

$$= \frac{pn}{6^2}\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{\frac{p}{3}n}-1\right]^2$$

$$\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}n}+1\right]^2\left[\left(\sqrt{2}+1\right)^{\frac{p}{6}n}+\left(\sqrt{2}-1\right)^{\frac{p}{6}n}\right]^2,$$

and

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{3}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{3}n,\frac{p}{2}n}\right)} = \left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{2p}{3}}\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{2p}{3}}\left(\sqrt{2}+1\right)^{\frac{p}{3}}.$$

From the asymptotic numbers of $T\left(C_{pn}^{1,bn,cn}\right)$, we see easily that in the graphs $C_{pn}^{1,bn,cn}$ satisfying $\frac{p}{b}, \frac{p}{c} \in \{2,3,4,6\}$ the asymptotic value attains its maximum when $\frac{p}{b}=6$, $\frac{p}{c}=4$.

(III) The case $T\left(C_{pn}^{1,bn,cn,dn}\right)$, where $\frac{p}{b}, \frac{p}{c}, \frac{p}{d} \in \{2,3,4,6\}$.

(k) If $\frac{p}{b}=6$, $\frac{p}{c}=4$, $\frac{p}{d}=3$, then

$$\begin{aligned}
T\left(C_{pn}^{1,bn,cn,dn}\right) &= \frac{1}{pn}\prod_{j=1}^{pn-1}\left[8-2\cos\frac{2\pi j}{pn}-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}\right] \\
&= \frac{1}{pn}\prod_{j=1}^{pn-1}\left(8-2\cos\frac{2\pi j}{pn}\right)\prod_{j=1}^{\frac{p}{6}n-1}\frac{\left(6-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}{\left(12-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}\prod_{j=1}^{\frac{p}{12}n-1}\frac{\left(2-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)\left(12-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)}{\left(6-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)\left(8-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)} \\
&\quad\times\prod_{j=1}^{\frac{p}{2}n-1}\frac{\left(12-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}{\left(8-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}\prod_{j=1}^{\frac{p}{4}n-1}\frac{\left(8-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)}{\left(12-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)} \\
&= \frac{1}{pn}U_{pn-1}^2\left(\sqrt{\frac{8+2}{4}}\right)\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{12+2}{4}}\right)}\frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{2+2}{4}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}\frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{12+2}{4}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{8+2}{4}}\right)} \\
&\quad\times\frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{12+2}{4}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{8+2}{4}}\right)}\frac{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{8+2}{4}}\right)}{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{12+2}{4}}\right)} \\
&= \frac{1}{pn}U_{pn-1}^2\left(\sqrt{\frac{5}{2}}\right)\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{2}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{7}{2}}\right)}\frac{U_{\frac{p}{12}n-1}^2(1)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{2}\right)}\frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{7}{2}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{5}{2}}\right)}\frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{7}{2}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{5}{2}}\right)}\frac{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{5}{2}}\right)}{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{7}{2}}\right)} \\
&= \frac{pn}{12^2}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{2}n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{2}n}\right]^2\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n}+1\right]^2 \\
&\quad\times\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{\frac{p}{6}n}-1\right]^2\left[\left(\sqrt{2}+1\right)^{\frac{p}{12}n}+\left(\sqrt{2}-1\right)^{\frac{p}{12}n}\right]^2,
\end{aligned}$$

and therefore

$$\lim_{n\to\infty}\frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{4}(n+1),\frac{p}{3}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{4}n,\frac{p}{3}n}\right)}=\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{4p}{3}}\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{p}{3}}\left(\sqrt{2}+1\right)^{\frac{p}{6}}.$$

(l) If $\frac{p}{b}=6$, $\frac{p}{c}=4$, $\frac{p}{d}=2$, then

$$\begin{aligned}
T\left(C_{pn}^{1,bn,cn,dn}\right) &= \frac{1}{pn}\prod_{j=1}^{pn-1}\left[8-2\cos\frac{2\pi j}{pn}-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}-2\cos\pi j\right] \\
&= \frac{1}{pn}\prod_{j=1}^{pn-1}\left(9-2\cos\frac{2\pi j}{pn}\right)\prod_{j=1}^{\frac{p}{6}n-1}\frac{\left(6-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}{\left(12-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}\prod_{j=1}^{\frac{p}{3}n-1}\frac{\left(12-2\cos\frac{2\pi j}{\frac{p}{3}n}\right)}{\left(9-2\cos\frac{2\pi j}{\frac{p}{3}n}\right)} \\
&\quad\times\prod_{j=1}^{\frac{p}{4}n-1}\frac{\left(5-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)}{\left(9-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)}\prod_{j=1}^{\frac{p}{12}n-1}\frac{\left(2-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)\left(9-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)}{\left(6-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)\left(5-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)} \\
&= \frac{1}{pn}U_{pn-1}^2\left(\sqrt{\frac{9+2}{4}}\right)\frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{12+2}{4}}\right)}\frac{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{12+2}{4}}\right)}{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)} \\
&\quad\times\frac{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}\frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{2+2}{4}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}\frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}
\end{aligned}$$

$$= \frac{1}{pn} U^2_{pn-1}\left(\sqrt{\frac{11}{4}}\right) \frac{U^2_{\frac{p}{6}n-1}\left(\sqrt{2}\right)}{U^2_{\frac{p}{6}n-1}\left(\sqrt{\frac{7}{2}}\right)} \frac{U^2_{\frac{p}{3}n-1}\left(\sqrt{\frac{7}{2}}\right)}{U^2_{\frac{p}{3}n-1}\left(\sqrt{\frac{11}{4}}\right)} \frac{U^2_{\frac{p}{4}n-1}\left(\sqrt{\frac{7}{4}}\right)}{U^2_{\frac{p}{4}n-1}\left(\sqrt{\frac{11}{4}}\right)} \frac{U^2_{\frac{p}{12}n-1}(1)}{U^2_{\frac{p}{12}n-1}\left(\sqrt{2}\right)} \frac{U^2_{\frac{p}{12}n-1}\left(\sqrt{\frac{11}{4}}\right)}{U^2_{\frac{p}{12}n-1}\left(\sqrt{\frac{7}{4}}\right)}$$

$$= \frac{pn}{12^2}\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{2p}{3}n} + \left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{\frac{2p}{3}n} + 1\right]^2$$

$$\times \left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n} + \left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n} + 1\right]^{-2} \left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{p}{6}n} + \left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{\frac{p}{6}n}\right]^2$$

$$\times \left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n} + \left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n} + 1\right]^2 \left[\left(\sqrt{2}+1\right)^{\frac{p}{12}n} + \left(\sqrt{2}-1\right)^{\frac{p}{12}n}\right]^2,$$

and

$$\lim_{n\to\infty} \frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{4}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{4}n,\frac{p}{2}n}\right)} = \left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{p} \left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{p}{3}} \left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}} \left(\sqrt{2}+1\right)^{\frac{p}{6}}.$$

(m) If $\frac{p}{b} = 6$, $\frac{p}{c} = 3$, $\frac{p}{d} = 2$, then

$$T\left(C_{pn}^{1,bn,cn,dn}\right) = \frac{1}{pn}\prod_{j=1}^{pn-1}\left[8 - 2\cos\frac{2\pi j}{pn} - 2\cos\frac{\pi j}{3} - 2\cos\frac{2\pi j}{3} - 2\cos\pi j\right]$$

$$= \frac{1}{pn}\prod_{j=1}^{pn-1}\left(10 - 2\cos\frac{2\pi j}{pn}\right)\prod_{j=1}^{\frac{p}{2}n-1}\frac{(8-2\cos\frac{2\pi j}{\frac{p}{2}n})}{(10-2\cos\frac{2\pi j}{\frac{p}{2}n})}\prod_{j=1}^{\frac{p}{6}n-1}\frac{(2-2\cos\frac{2\pi j}{\frac{p}{6}n})}{(8-2\cos\frac{2\pi j}{\frac{p}{6}n})}$$

$$= \frac{1}{pn}U^2_{pn-1}\left(\sqrt{\frac{10+2}{4}}\right)\frac{U^2_{\frac{p}{2}n-1}\left(\sqrt{\frac{8+2}{4}}\right)}{U^2_{\frac{p}{2}n-1}\left(\sqrt{\frac{10+2}{4}}\right)}\frac{U^2_{\frac{p}{6}n-1}\left(\sqrt{\frac{2+2}{4}}\right)}{U^2_{\frac{p}{6}n-1}\left(\sqrt{\frac{8+2}{4}}\right)}$$

$$= \frac{1}{pn}U^2_{pn-1}(\sqrt{3})\frac{U^2_{\frac{p}{2}n-1}\left(\sqrt{\frac{5}{2}}\right)}{U^2_{\frac{p}{2}n-1}(\sqrt{3})}\frac{U^2_{\frac{p}{6}n-1}(1)}{U^2_{\frac{p}{6}n-1}\left(\sqrt{\frac{5}{2}}\right)}$$

$$= \frac{pn}{6^2}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}n} + \left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}n} + 1\right]^2 \left[\left(\sqrt{3}+\sqrt{2}\right)^{\frac{p}{2}n} + \left(\sqrt{3}-\sqrt{2}\right)^{\frac{p}{2}n}\right]^2,$$

implying that

$$\lim_{n\to\infty} \frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{3}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{3}n,\frac{p}{2}n}\right)} = \left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{2p}{3}} \left(\sqrt{3}+\sqrt{2}\right)^{p}.$$

(n) If $\frac{p}{b} = 4$, $\frac{p}{c} = 3$, $\frac{p}{d} = 2$, then

$$T\left(C_{pn}^{1,bn,cn,dn}\right) = \frac{1}{pn}\prod_{j=1}^{pn-1}\left[8 - 2\cos\frac{2\pi j}{pn} - 2\cos\frac{\pi j}{2} - 2\cos\frac{2\pi j}{3} - 2\cos\pi j\right]$$

$$= \frac{1}{pn}\prod_{j=1}^{pn-1}\left(11 - 2\cos\frac{2\pi j}{pn}\right)\prod_{j=1}^{\frac{p}{3}n-1}\frac{\left(8-2\cos\frac{2\pi j}{\frac{p}{3}n}\right)}{\left(11-2\cos\frac{2\pi j}{\frac{p}{3}n}\right)}\prod_{j=1}^{\frac{p}{2}n-1}\frac{\left(9-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}{\left(11-2\cos\frac{2\pi j}{\frac{p}{2}n}\right)}$$

$$
\begin{aligned}
&\times \prod_{j=1}^{\frac{p}{6}n-1} \frac{\left(6-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)\left(11-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}{\left(8-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)\left(9-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)} \prod_{j=1}^{\frac{p}{4}n-1} \frac{\left(5-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)}{\left(9-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)} \\
&\times \prod_{j=1}^{\frac{p}{12}n-1} \frac{\left(2-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)\left(9-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)}{\left(6-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)\left(5-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)} \\
&= \frac{1}{pn} U_{pn-1}^2\left(\sqrt{\frac{11+2}{4}}\right) \frac{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{8+2}{4}}\right)}{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{11+2}{4}}\right)} \frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{11+2}{4}}\right)} \frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{8+2}{4}}\right)} \frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{11+2}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)} \\
&\times \frac{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)}{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)} \frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{2+2}{4}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{6+2}{4}}\right)} \frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{9+2}{4}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{5+2}{4}}\right)} \\
&= \frac{1}{pn} U_{pn-1}^2\left(\sqrt{\frac{13}{4}}\right) \frac{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{5}{2}}\right)}{U_{\frac{p}{3}n-1}^2\left(\sqrt{\frac{13}{4}}\right)} \frac{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{11}{4}}\right)}{U_{\frac{p}{2}n-1}^2\left(\sqrt{\frac{13}{4}}\right)} \frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{2}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{5}{2}}\right)} \frac{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{13}{4}}\right)}{U_{\frac{p}{6}n-1}^2\left(\sqrt{\frac{11}{4}}\right)} \\
&\times \frac{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{7}{4}}\right)}{U_{\frac{p}{4}n-1}^2\left(\sqrt{\frac{11}{4}}\right)} \frac{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{11}{4}}\right)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{\frac{7}{4}}\right)} \frac{U_{\frac{p}{12}n-1}^2(1)}{U_{\frac{p}{12}n-1}^2\left(\sqrt{2}\right)} \\
&= \frac{pn}{12^2}\left[\left(\sqrt{\frac{13}{4}}+\sqrt{\frac{9}{4}}\right)^{\frac{p}{3}n}+\left(\sqrt{\frac{13}{4}}-\sqrt{\frac{9}{4}}\right)^{\frac{p}{3}n}-1\right]^2 \\
&\times \left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{\frac{p}{6}n}-1\right]^2 \left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{\frac{p}{6}n}+1\right]^2 \\
&\times \left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n}\right]^2 \left[\left(\sqrt{2}+1\right)^{\frac{p}{12}n}+\left(\sqrt{2}-1\right)^{\frac{p}{12}n}\right]^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{T\left(C_{p(n+1)}^{1,\frac{p}{4}(n+1),\frac{p}{3}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{4}n,\frac{p}{3}n,\frac{p}{2}n}\right)} \\
&= \left(\sqrt{\frac{13}{4}}+\sqrt{\frac{9}{4}}\right)^{\frac{2p}{3}} \left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{\frac{p}{3}} \left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}} \left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{\frac{p}{3}} \left(\sqrt{2}+1\right)^{\frac{p}{6}}.
\end{aligned}
$$

From the asymptotics of $T\left(C_{pn}^{1,bn,cn,dn}\right)$, one sees that in graphs $C_{pn}^{1,bn,cn,dn}$ satisfying $\frac{p}{b}, \frac{p}{c}, \frac{p}{d} \in \{2, 3, 4, 6\}$, the maximum value attains when $\frac{p}{b} = 6$, $\frac{p}{c} = 4$, $\frac{p}{d} = 3$.

(IV) The case $T\left(C_{pn}^{1,bn,cn,dn,en}\right)$, where $\frac{p}{b}, \frac{p}{c}, \frac{p}{d}, \frac{p}{e} \in \{2, 3, 4, 6\}$.

(o) If $\frac{p}{b} = 6$, $\frac{p}{c} = 4$, $\frac{p}{d} = 3$, $\frac{p}{e} = 2$, then, similarly,

$$
\begin{aligned}
T\left(C_{pn}^{1,bn,cn,dn,en}\right) &= \frac{1}{pn}\prod_{j=1}^{pn-1}\left[10-2\cos\frac{2\pi j}{pn}-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}-2\cos\pi j\right] \\
&= \frac{1}{pn}\prod_{j=1}^{pn-1}\left(12-2\cos\frac{2\pi j}{pn}\right)\prod_{j=1}^{\frac{p}{6}n-1}\frac{\left(6-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}{\left(12-2\cos\frac{2\pi j}{\frac{p}{6}n}\right)}
\end{aligned}
$$

$$\times \prod_{j=1}^{\frac{p}{4}n-1} \frac{\left(8-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)}{\left(12-2\cos\frac{2\pi j}{\frac{p}{4}n}\right)} \prod_{j=1}^{\frac{p}{12}n-1} \frac{\left(2-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)\left(12-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)}{\left(6-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)\left(8-2\cos\frac{2\pi j}{\frac{p}{12}n}\right)}$$

$$= \frac{1}{pn} U^2_{pn-1}\left(\sqrt{\frac{12+2}{4}}\right) \frac{U^2_{\frac{p}{6}n-1}\left(\sqrt{\frac{6+2}{4}}\right)}{U^2_{\frac{p}{6}n-1}\left(\sqrt{\frac{12+2}{4}}\right)} \frac{U^2_{\frac{p}{4}n-1}\left(\sqrt{\frac{8+2}{4}}\right)}{U^2_{\frac{p}{4}n-1}\left(\sqrt{\frac{12+2}{4}}\right)} \frac{U^2_{\frac{p}{12}n-1}\left(\sqrt{\frac{2+2}{4}}\right)}{U^2_{\frac{p}{12}n-1}\left(\sqrt{\frac{6+2}{4}}\right)} \frac{U^2_{\frac{p}{12}n-1}\left(\sqrt{\frac{12+2}{4}}\right)}{U^2_{\frac{p}{12}n-1}\left(\sqrt{\frac{8+2}{4}}\right)}$$

$$= \frac{1}{pn} U^2_{pn-1}\left(\sqrt{\frac{7}{2}}\right) \frac{U^2_{\frac{p}{6}n-1}\left(\sqrt{2}\right)}{U^2_{\frac{p}{6}n-1}\left(\sqrt{\frac{7}{2}}\right)} \frac{U^2_{\frac{p}{4}n-1}\left(\sqrt{\frac{5}{2}}\right)}{U^2_{\frac{p}{4}n-1}\left(\sqrt{\frac{7}{2}}\right)} \frac{U^2_{\frac{p}{12}n-1}\left(\sqrt{\frac{7}{2}}\right)}{U^2_{\frac{p}{12}n-1}\left(\sqrt{\frac{5}{2}}\right)} \frac{U^2_{\frac{p}{12}n-1}(1)}{U^2_{\frac{p}{12}n-1}\left(\sqrt{2}\right)}$$

$$= \frac{pn}{12^2}\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{p}{2}n} + \left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{\frac{p}{2}n}\right]^2 \left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{p}{6}n} + \left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{\frac{p}{6}n} - 1\right]^2$$

$$\times \left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n} + \left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{\frac{p}{6}n} + 1\right]^2 \left[\left(\sqrt{2}+1\right)^{\frac{p}{12}n} + \left(\sqrt{2}-1\right)^{\frac{p}{12}n}\right]^2,$$

thus

$$\lim_{n\to\infty} \frac{T\left(C_{p(n+1)}^{1,\frac{p}{6}(n+1),\frac{p}{4}(n+1),\frac{p}{3}(n+1),\frac{p}{2}(n+1)}\right)}{T\left(C_{pn}^{1,\frac{p}{6}n,\frac{p}{4}n,\frac{p}{3}n,\frac{p}{2}n}\right)} = \left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{\frac{4p}{3}} \left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{\frac{p}{3}} \left(\sqrt{2}+1\right)^{\frac{p}{6}}.$$

References

[1] R. Bapat, I. Gutman, W. Xiao, A simple method for computing resistance distance, Z. Naturforsch 58a (2003) 494–498.
[2] N. Biggs, Algebraic Graph Theory, second ed. Cambridge University Press, London, 1993.
[3] F.T. Boesch, H. Prodinger, Spanning tree formulas and Chebyshev polynomials, Graphs Combin. 2 (1986) 191–200.
[4] F.T. Boesch, J.F. Wang, A conjecture on the number of spanning trees in the square of a cycle, in: Notes from New York Graph Theory Day V, page 16, 1982.
[5] X. Chen, An asymptotic enumeration theorem for the number of spanning trees in grids and tori, J. Zhangzhou Teach. Coll. Nat. Sci. 14 (2) (2001) 7–12 (in Chinese).
[6] X. Chen, The number of spanning trees in directed circulant graphs with non-fixed jumps, Discrete Math. 307 (2007) 1873–1880.
[7] D. Cvetkovič, M. Doob, H. Sachs, Spectra of Graphs: Theory and Applications, third ed. Johann Ambrosius Barth, Heidelberg, 1995.
[8] A. Garcia, M. Noy, J. Tejel, The asymptotic number of spanning trees in d-dimensional square lattices, J. Combin. Math. Combin. Comput. 44 (2003) 109–113.
[9] M.J. Golin, Y. Leung, Y. Wang, X. Yong, Counting structures in grid graphs, cylinders and tori using transfer matrices: survey and new results, in: The Proceedings of SIAM ALENEX/ANALCO (Workshop on Analytic Algorithms and Combinatorics), British Columbia, Canada, 2005.
[10] M.J. Golin, X. Yong, Y. Zhang, The asymptotic number of spanning trees in circulant graphs, Discrete Math. 310 (4) (2010) 792–803.
[11] G. Kirchhoff, Über die Auflösung der gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird, Annu. Rev. Phys. Chem. 72 (1847) 497–508.
[12] Z. Lonc, K. Parol, J.M. Wojciechowski, On the asymptotic behavior of the maximum number of spanning trees in circulant graphs, Networks 30 (1) (1997) 47–56.
[13] Z. Lonc, K. Parol, J.M. Wojciechowski, On the number of spanning trees in directed circulant graphs, Networks 37 (3) (2001) 129–133.
[14] J. Louis, A formula for the number of spanning trees in circulant graphs with non-fixed generators, Manuscript.
[15] R. Lyons, Asymptotic enumeration of spanning trees, Combin. Probab. Comput. 14 (4) (2005) 491–522.
[16] S. Seshu, M.B. Reed, Linear Graphs and Electrical Networks, Addison-Wesley, Reading, MA, 1961.
[17] Daniel A. Spielman, Algorithms, Graph Theory, and Linear Equations in Laplacian Matrices, in: Proceedings of the International Congress of Mathematicians Hyderabad, India, 2010. Or see: http://www.cs.yale.edu/homes/spielman/PAPERS/icm10post.pdf.
[18] X. Yong, Talip, Acenjian, The numbers of spanning trees of the cubic cycle C_N^3 and the quadruple cycle C_N^4, Discrete Math. 169 (1997) 293–298.
[19] F. Zhang, X. Yong, Asymptotic enumeration theorems for the numbers of spanning trees and Eulerian trails in circulant digraphs & graphs, Science in China (Series A) 43 (3) (1999) 264–271.
[20] Y. Zhang, X. Yong, M. Golin, The number of spanning trees in circulant graphs, Discrete Math. 223 (2000) 337–350.
[21] Y. Zhang, X. Yong, M. Golin, Chebyshev polynomials and spanning tree formulas for circulant and related graphs, Discrete Math. 298 (2005) 334–364.