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On the spectral characterization of some unicyclic graphs

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ABSTRACT

Let $H(n; q, n_1, n_2)$ be a graph with n vertices containing a cycle C_q and two hanging paths P_{n_1} and P_{n_2} attached at the same vertex of the cycle. In this paper, we prove that except for the A-cospectral graphs $H(12; 6, 1, 5)$ and $H(12; 8, 2, 2)$, no two non-isomorphic graphs of the form $H(n; q, n_1, n_2)$ are A-cospectral. It is proved that all graphs $H(n; q, n_1, n_2)$ are determined by their L-spectra. And all graphs $H(n; q, n_1, n_2)$ are proved to be determined by their Q-spectra, except for graphs $H(2a+4; a+3, \frac{a}{2}, \frac{a}{2}+1)$ with a being a positive even number and $H(2b; b, \frac{b}{2}, \frac{b}{2})$ with $b \geq 4$ being an even number. Moreover, the Q-cospectral graphs with these two exceptions are given.

1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $v_1, v_2, \ldots, v_n$ are indexed in the non-increasing order of degrees. All graphs considered here are simple and undirected. Let matrix $A(G)$ be the $(0, 1)$-*adjacency matrix* of G and $d_i = d_i(G) = d_G(v_i)$ the degree of the vertex v_i. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G, where $D(G)$ is the $n \times n$ diagonal matrix with $\{d_1, d_2, \ldots, d_n\}$ as diagonal entries. The matrix $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* of G. The polynomials $P_{A(G)}(\lambda) = \det(\lambda I - A(G))$, $P_{L(G)}(\mu) = \det(\mu I - L(G))$ and $P_{Q(G)}(\mu) = \det(\mu I - Q(G))$, where I is the identity matrix, are defined as the characteristic polynomials of the graph G with respect to the adjacency matrix, the Laplacian matrix and the signless Laplacian matrix, respectively, which can be written as $P_{A(G)}(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$, $P_{L(G)}(\mu) = l_0\mu^n + l_1\mu^{n-1} + \cdots + l_n$ and $P_{Q(G)}(\nu) = q_0\nu^n + q_1\nu^{n-1} + \cdots + q_n$. Since matrices $A(G)$, $L(G)$ and $Q(G)$ are real and symmetric, their eigenvalues are all real numbers. Assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ are respectively the *adjacency eigenvalues*, the *Laplacian eigenvalues* and the *signless Laplacian eigenvalues* of graph G. The *A-spectrum* (or *L-spectrum*, *Q-spectrum*) of the graph G consists of the adjacency eigenvalues (or Laplacian eigenvalues, signless Laplacian eigenvalues). Two graphs G and H are said to be *A-cospectral* (or *L-cospectral*, *Q-cospectral*) if they have equal A-spectrum (or L-spectrum, Q-spectrum) [1]. A graph is said to be determined by the A-spectrum (or L-spectrum, Q-spectrum) if there is no other non-isomorphic graph with the same A-spectrum (or L-spectrum, Q-spectrum).

Characterizing the graphs that are determined by their spectra seems to be a difficult problem in the theory of graph spectra. Up until now, many graphs have been proved to be determined by their spectra. The readers can consult [11,16,20,19,23,24,26,25].

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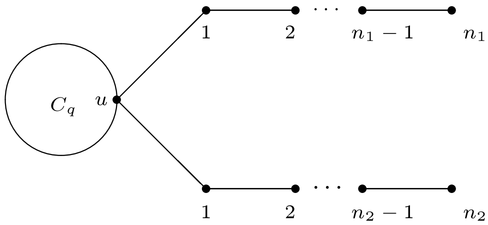

Fig. 1. Graph $H(n; q, n_1, n_2)$.

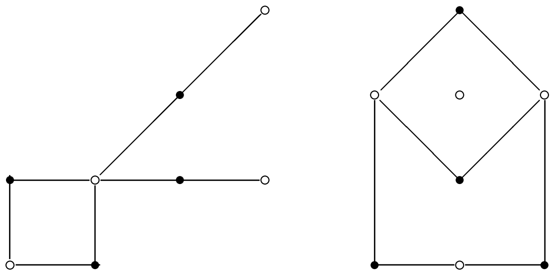

Fig. 2. The A-cospectral graphs $H(8; 4, 2, 2)$ and G'.

In this paper, we will characterize the graph $H(n; q, n_1, n_2)$ (shown in Fig. 1) by its spectra, which contains a cycle C_q and two hanging paths P_{n_1} and P_{n_2} attached at the same vertex of the cycle. Note that if we append a pendant vertex of a path P_k to a cycle C_q, it is just the lollipop graph $L(q, k)$ [2,11]. In [11], the lollipop graph with q odd is proved to be determined by its A-spectrum, and all lollipop graphs are proved to be determined by their L-spectra. Also the lollipop graphs with an even cycle are proved to be determined by their A-spectra (Tayfeh-Rezaie [private communication] did the lollipop graphs with a cycle of length at least 6, and Boulet and Jouve [2] did the general case). Whether all graphs $H(n; q, n_1, n_2)$ are determined by their A-spectra? Unfortunately, the answer is negative. By Godsil–McKay switching [10], graph G' which is A-cospectral to graph $H(8; 4, 2, 2)$ is found out (see Fig. 2), and A-cospectral graphs $H(12; 6, 1, 5)$ and $H(12; 8, 2, 2)$ are also found out in Theorem 3.4. In Section 5, A-cospectral graphs $H(n; 2a + 6, a, a + 2)$ and $\Lambda(a, a, 2a + 2)$ with a being a positive even number and A-cospectral graphs $H(n; 2b, b, b)$ and $\Theta(b - 2, 2b - 3, b - 1)$ with $b \geq 4$ being a positive even number are found out in Lemmas 5.8 and 5.11, respectively.

This paper is organized as follows. In Section 2, some available lemmas are summarized. In Section 3, it is proved that except for the A-cospectral graphs $H(12; 6, 1, 5)$ and $H(12; 8, 2, 2)$, no two non-isomorphic graphs of the form $H(n; q, n_1, n_2)$ are A-cospectral. In Section 4, $H(n; q, n_1, n_2)$ is proved to be determined by its L-spectrum. In Section 5, it is proved that all graphs $H(n; q, n_1, n_2)$ are determined by their Q-spectra, except for graphs $H(2a + 4; a + 3, \frac{a}{2}, \frac{a}{2} + 1)$ with a being a positive even number and $H(2b; b, \frac{b}{2}, \frac{b}{2})$ with $b \geq 4$ being an even number.

2. Preliminaries

Some previously established results about the spectrum are summarized in this section. They will play important roles throughout the paper.

Lemma 2.1 ([5]). *Let u be a vertex of G, $N(u)$ be the set of all vertices adjacent to u and $C(u)$ be the set of all cycles containing u. The characteristic polynomial of G satisfies*

$$P_{A(G)}(\lambda) = \lambda P_{A(G-u)}(\lambda) - \sum_{v \in N(u)} P_{A(G-u-v)}(\lambda) - 2 \sum_{Z \in C(u)} P_{A(G \setminus V(Z))}(\lambda).$$

Some results of [18,23] are summarized in the following lemma.

Lemma 2.2. *Let G be a graph. For the adjacency matrix and the Laplacian matrix, the following can be deduced from the spectrum:*

(i) *The number of vertices.*
(ii) *The number of edges.*
(iii) *Whether G is regular.*
(iv) *Whether G is regular with any fixed girth.*
For the adjacency matrix, the following follows from the spectrum.
(v) *The number of closed walk of any length.*

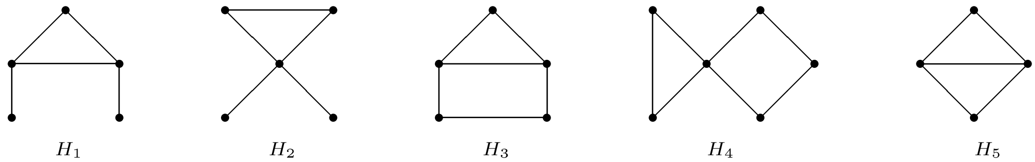

Fig. 3. Graphs H_i for $i = 1, 2, \ldots, 5$.

(vi) *Whether G is bipartite.*

For the Laplacian matrix, the following follows from the spectrum.

(vii) *The number of components.*
(viii) *The number of spanning trees.*
(ix) *The sum of the squares of degrees of vertices.*

Lemma 2.3 ([6]). *Let G be a graph with n vertices and m edges, and $n_G(C_4)$ is the number of subgraph C_4. Let x_k be the number of vertices of degree k in G. Then we have*

$$\sum_i \lambda_i^4 = 8n_G(C_4) + \sum_k kx_k + 4\sum_{k\geq 2}\frac{k(k-1)}{2}x_k.$$

Lemma 2.4 ([19]). *Let G be a graph and $N_G(i)$ the number of closed walks of length i in G, then*

$$N_G(5) = 30n_G(K_3) + 10n_G(C_5) + 10n_G(L(3,1)),$$
$$N_G(7) = 126n_G(K_3) + 84n_G(L(3,1)) + 14n_G(H_1) + 14n_G(L(3,2)) + 14n_G(L(5,1)) + 28n_G(H_2) + 42n_G(H_3) + 28n_G(H_4) + 112n_G(H_5) + 70n_G(C_5) + 14n_G(C_7),$$

where $L(q, k)$ is the lollipop graph, and graphs H_i for $i = 1, 2, \ldots, 5$ are shown in Fig. 3.

Lemma 2.5 ([18]). *Let G be a graph with n vertices and m edges and let $\deg(G) = (d_1, d_2, \ldots, d_n)$ be its non-increasing degree sequence. Then the first four coefficients in $P_{L(G)}(\mu)$ are*

$$l_0 = 1, \qquad l_1 = -2m, \qquad l_2 = 2m^2 - m - \frac{1}{2}\sum_{i=1}^n d_i^2,$$
$$l_3 = \frac{1}{3}\left(-4m^3 + 6m^2 + 3m\sum_{i=1}^n d_i^2 - \sum_{i=1}^n d_i^3 - 3\sum_{i=1}^n d_i^2 + 6n_G(C_3)\right).$$

Lemma 2.6 ([14,17]). *Let G be a graph with $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$. Then*

$$d_1 + 1 \leq \mu_1 \leq max\left\{\frac{d_i(d_i+m_i) + d_j(d_j+m_j)}{d_i + d_j},\ v_iv_j \in E(G)\right\},$$

where m_i denotes the average of the degrees of the vertices adjacent to vertex v_i in G.

Lemma 2.7 ([15]). *Let G be a connected graph with $n \geq 3$ vertices. Then $\mu_2 \geq d_2$.*

Lemma 2.8 ([7,21]). *Let G be a graph with n vertices, m edges, $n_G(C_3)$ triangles and $\deg(G) = (d_1, d_2, \ldots, d_n)$. Let $T_k = \sum_{i=1}^n v_i^k$, $(k = 0, 1, 2, \ldots)$ be the kth spectral moment for the Q-spectrum. Then*

$$T_0 = n, \qquad T_1 = \sum_{i=1}^n d_i = 2m, \qquad T_2 = 2m + \sum_{i=1}^n d_i^2, \qquad T_3 = 6n_G(C_3) + 3\sum_{i=1}^n d_i^2 + \sum_{i=1}^n d_i^3.$$

From Lemma 2.8, we can easily get the following result.

Lemma 2.9. *If H is a graph Q-cospectral to G, then*

(i) *G and H have the same number of vertices.*
(ii) *G and H have the same number of edges.*
(iii) $\sum_{i=1}^n d_i(G)^2 = \sum_{i=1}^n d_i(H)^2$.
(iv) $6n_G(C_3) + \sum_{i=1}^n d_i(G)^3 = 6n_H(C_3) + \sum_{i=1}^n d_i(H)^3$.

3. A-spectral characterization of graphs $H(n; q, n_1, n_2)$

First, we will prove that the two graphs in Fig. 2 and their complements are A-cospectral, respectively.

Theorem 3.1. *The graph $H(8; 4, 2, 2)$ and the graph G' given in Fig. 2 are A-cospectral. The same is true for their complements.*

Proof. Consider the four black vertices $H(8; 4, 2, 2)$ in Fig. 2. For each black vertex v, delete the edges between v and the white neighbors, and insert edges between v and the other white vertices. It is easily checked that this operation transforms $H(8; 4, 2, 2)$ into G'. Godsil and McKay (see [10], this operation is called Godsil–McKay switching) have shown that this operation leaves the A-spectrum of the graph and its complement unchanged. □

It is clear that $H(8; 4, 2, 2)$ and G' are non-isomorphic. So $H(8; 4, 2, 2)$ is not determined by its A-spectrum. Since also the complements of $H(8; 4, 2, 2)$ and G' are A-cospectral, it also follows that $H(8; 4, 2, 2)$ is not determined by the spectra of all its generalized adjacency matrices, where a generalized adjacency matrix (see [10]) is just a linear combination of matrices A, I and J (the all-ones matrix).

Corollary 3.2. *Graph $H(8; 4, 2, 2)$ is not determined by the spectra of all its generalized adjacency matrices.*

In the following, we show that except for the A-cospectral graphs $H(12; 6, 1, 5)$ and $H(12; 8, 2, 2)$, no two non-isomorphic graphs of the form $H(n; q, n_1, n_2)$ are A-cospectral. Indeed, A-cospectral graphs $H(12; 6, 1, 5)$ and $H(12; 8, 2, 2)$ can be figured out by the proof of Theorem 3.4 (see Case 8), and by Maple, they both have the adjacency characteristic polynomial: $\lambda^{12} - 12\lambda^{10} + 51\lambda^8 - 96\lambda^6 + 80\lambda^4 - 24\lambda^2$.

For the sake of simplicity and with a slight abuse of notation, let $\Gamma_1 = H(n; t + 1, s, k)$ and we denote $P_{A(P_n)}(\lambda)$ by $p_n = P_{A(P_n)}(\lambda)$. By convention, let $p_0 = 1$, $p_{-1} = 0$ and $p_{-2} = -1$. Using Lemma 2.1, with u being the vertex of degree 4 in Γ_1, we can compute the characteristic polynomial of Γ_1 in terms of the characteristic polynomials of paths as follows

$$P_{A(\Gamma_1)}(\lambda) = \lambda p_s p_k p_t - p_t p_{s-1} p_k - p_t p_s p_{k-1} - 2p_{t-1} p_s p_k - 2p_s p_k. \tag{3.1}$$

The next lemma follows from (3.1) and $p_r(2) = r + 1$.

Lemma 3.3. $P_{A(\Gamma_1)}(2) = -2skt - 2sk - st - kt - s - k$.

Note that by Lemma 2.1, we have $p_r = \lambda p_{r-1} - p_{r-2}$. Solving this recurrence equation, we find that for $r \geq -2$,

$$p_r = \frac{x^{2r+2} - 1}{x^{r+2} - x^r}, \tag{3.2}$$

where x satisfies $x^2 - \lambda x + 1 = 0$ with $\lambda \neq 2$. Substituting (3.2) in (3.1), by using Maple, we can obtain

$$x^n(x^2 - 1)^3 P_{A(\Gamma_1)}(\lambda) + 1 - 3x^2 - x^{2n+6} + 3x^{2n+4} = f(s, k, t; x), \tag{3.3}$$

where $n = s + k + t + 1$ and

$$\begin{aligned} f(s, k, t; x) = {} & 2x^{t+1} - 2x^{2k+4} - 2x^{2s+4} - 2x^{t+3} - x^{2t+2} - x^{2t+4} - 2x^{t+3+2s} + 2x^{t+5+2s} - 2x^{t+3+2k} + 2x^{t+5+2k} \\ & + x^{2s+4+2k} + x^{2s+6+2k} + 2x^{2s+4+2t} + 2x^{4+2k+2t} + 2x^{t+5+2s+2k} - 2x^{t+7+2s+2k}. \end{aligned}$$

Theorem 3.4. *Except for the graphs $H(12; 6, 1, 5)$ and $H(12; 8, 2, 2)$, no two non-isomorphic graphs of the form $H(n; t+1, s, k)$ are A-cospectral.*

Proof. Suppose that $H(n; t + 1, s, k)$ and $H(n'; t' + 1, s', k')$ are A-cospectral, then they have the same number of vertices, we have

$$t + s + k + 1 = t' + s' + k' + 1 = n = n'. \tag{3.4}$$

By Lemma 3.3,

$$2skt + 2sk + st + kt + s + k = 2s'k't' + 2s'k' + s't' + k't' + s' + k', \tag{3.5}$$

and by (3.3),

$$f(s, k, t; x) = f(s', k', t'; x). \tag{3.6}$$

Without loss of generality, we can assume $s \leq k$ and $s' \leq k'$. Now enumerate the different possibilities for the monomial of smallest degree of $f(s, k, t; x)$ (the same can be done for $f(s', k', t'; x)$):

(3.7)

- $2x^{t+1}$ if $t + 1 < 2s + 4$,
- $-2x^{2s+4}$ if $t + 1 > 2s + 4$ and $s \neq k$,
- $-4x^{2s+4}$ if $t + 1 > 2s + 4$ and $s = k$,
- $-2x^{2k+4}$ if $t + 1 = 2s + 4$ and $t + 3 > 2k + 4$ (that is $s = k$),
- $-2x^{t+3}$ if $t + 1 = 2s + 4$ and $t + 3 < 2k + 4$ (that is $k > s + 1$),
- $-4x^{t+3}$ if $t + 1 = 2s + 4$ and $t + 3 = 2k + 4$ (that is $k = s + 1$).

Therefore, by exchanging the roles of $f(s,k,t;x)$ and $f(s',k',t';x)$ and by (3.6), it suffices to consider the following cases:

Case 1. $2x^{t+1}=2x^{t'+1}$ (the hypotheses are $t+1<2s+4$ and $t'+1<2s'+4$). Then $t=t'$. By (3.4) and (3.5), we obtain that $s+k=s'+k'$ and $sk=s'k'$, which leads to $s=k$ and $s'=k'$ or $s=k'$ and $s'=k$.

Case 2. $-2x^{2s+4}=-2x^{2s'+4}$ (the hypotheses are $t+1>2s+4,\ s\neq k$ and $t'+1>2s'+4,\ s'\neq k'$). Then $s=s'$. Let $k'=k-i$ with i an integer. By (3.4), $s=s'$ implies that $t'=t+i$. Suppose $i\neq 0$. Expressing s',k' and t' by s,k,t and i in (3.5), we get $(2s+1)(k-t-i)=s+1$. This is a contradiction, since $s+1<2s+1$. Then $i=0$, i.e., $k=k'$ and $t=t'$.

Case 3. $-2x^{2s+4}=-2x^{2k'+4}$ (the hypotheses are $t+1>2s+4,\ s\neq k$ and $t'+1=2s'+4,\ s'=k'$). Then $s=k'$. Using the similar computation as in Case 2, we obtain that $s'=k$ and $t=t'$. This means that $t+1=2s+4$, a contradiction to $t+1>2s+4$.

Case 4. $-2x^{2s+4}=-2x^{t'+3}$ (the hypotheses are $t+1>2s+4,\ s\neq k$ and $t'+1=2s'+4,\ k'>s'+1$). Then $t'=2s+1$. In this case, $t'+1=2s'+4$ and $t'=2s+1$ imply that $s'=s-1$. Then $s\geq 2$, since $s'\geq 1$. Substitute $t'=2s+1$ and $s'=s-1$ into $f(s',k',t';x)$, we transform $f(s',k',t';x)$ into

$$f(s',k',t';x)=-2x^{2s+4}-2x^{2k'+4}-2x^{4s+2}+x^{4s+4}-x^{4s+6}+x^{2s+2+2k'}$$
$$-x^{2s+4+2k'}+2x^{2s+6+2k'}+2x^{6s+4}+2x^{4s+4+2k'}. \tag{3.8}$$

Now, the different possibilities for the monomial of second smallest degree of $f(s',k',t';x)$ are

- $-2x^{2k'+4}$ if $2k'+4<4s+2$,
- $-2x^{4s+2}$ if $2k'+4>4s+2$,
- $-4x^{4s+2}$ if $2k'+4=4s+2$.

and the different possibilities for the monomial of second smallest degree of $f(s,k,t;x)$ are

- $-2x^{2k+4}$ if $2k+4<t+1$,
- $2x^{t+1}$ if $2k+4>t+1$,
- $-2x^{t+3}$ if $2k+4=t+1$.

Then consider the following subcases:

Case 4.1. $-2x^{2k'+4}=-2x^{2k+4}$ (the hypotheses are $t+1>2k+4>2s+4$, and $t'+1=2s'+4,\ k'>s'+1,\ 2k'+4<4s+2$). Then $k=k'$. Using the similar computation as in Case 2, we obtain that $s=s'$ and $t=t'$. This means that $t+1=2s+4$, a contradiction to $t+1>2s+4$.

Case 4.2. $-2x^{2k'+4}=-2x^{t+3}$ (the hypotheses are $t+1=2k+4>2s+4$, and $t'+1=2s'+4,\ k'>s'+1,\ 2k'+4<4s+2$). Then $t=2k'+1=2k+3$ and $t'=2s'+3=2s+1$, that is $k'=k+1$ and $s=s'+1$. Together with (3.4), we get $t=t'$. Then $t+1=2s+2<2s+4$, a contradiction to $t+1>2s+4$.

Case 4.3. $-2x^{4s+2}=-2x^{2k+4}$ (the hypotheses are $t+1>2k+4>2s+4$, and $t'+1=2s'+4,\ k'>s'+1,\ 2k'+4>4s+2$). Then $k=2s-1$ and $t'=2s+1=2s'+3$, that is $s=s'+1$. Together with (3.4), we get $t=k'+1$. Substitute $s,k=2s-1$, $t=k'+1, s'=s-1, k'$ and $t'=2s+1$ into (3.5) and by simplifying, we get $k'=6s+8+\frac{8}{s-1}$. Since k' is a positive integer, $s=2,\ 3,\ 5$ or 9. Then we get the following four cases:

$$\begin{cases} s=2, k=3, t=29,\\ s'=1, k'=28, t'=5.\end{cases} \qquad \begin{cases} s=3, k=5, t=31,\\ s'=2, k'=30, t'=7.\end{cases}$$

$$\begin{cases} s=5, k=9, t=41,\\ s'=4, k'=40, t'=11.\end{cases} \qquad \begin{cases} s=9, k=17, t=64,\\ s'=8, k'=63, t'=19.\end{cases}$$

Substitute them back into $f(s,k,t;x)$ and $f(s',k',t';x)$, respectively, and by simple computation, we always get $f(s,k,t;x)\neq f(s',k',t';x)$, contradictions.

Case 4.4. $-2x^{4s+2}=-2x^{t+3}$ (the hypotheses are $t+1=2k+4>2s+4$, and $t'+1=2s'+4,\ k'>s'+1,\ 2k'+4>4s+2$). Then $t=4s-1=2k+3$ and $t'=2s'+3=2s+1$, that is $k=2s-2$ and $s'=s-1$. Then by (3.4), we get that $k'=4s-3$. Substitute $s,k=2s-2,t=4s-1,s'=s-1,k'=4s-3$ and $t'=2s+1$ into (3.5) and by simplifying, we get $s^2-3s+2=0$. Then $s=2$, since $s\geq 2$. So $s=k=2$, a contradiction to $s<k$.

Case 5. $-4x^{2s+4}=-4x^{2s'+4}$ (the hypotheses are $t+1>2s+4,\ s=k$ and $t'+1>2s'+4,\ s'=k'$). Then $s=s'=k=k'$. Together with (3.4), we obtain that $t=t'$.

Case 6. $-4x^{2s+4}=-4x^{t'+3}$ (the hypotheses are $t+1>2s+4,\ s=k$ and $t'+1=2s'+4,\ k'=s'+1$). Then $t'=2s+1=2s'+3$, that is $k=s=s'+1=k'$. Using the similar computation as in Case 2, we obtain that $s=s'$, which is a contradiction to $s=s'+1$.

Case 7. $-2x^{2k+4} = -2x^{2k'+4}$ (the hypotheses are $t+1 = 2s+4$, $s = k$ and $t'+1 = 2s'+4$, $s' = k'$). Then $k = k' = s' = s = t = t'$.

Case 8. $-2x^{2k+4} = -2x^{t'+3}$ (the hypotheses are $t+1 = 2s+4$, $s = k$ and $t'+1 = 2s'+4$, $k' > s'+1$). Then $t' = 2k+1 = 2s'+3 = 2s+1$ and $t = 2s+3 = 2k+3$, that is $s' = k-1$. Then by (3.4), we get that $k' = k+3$. Substitute $s = k, t = 2k+3, s' = k-1, t' = 2k+1$ and $k' = k+3$ into (3.5), we obtain that $k^2 - k - 2 = 0$, that is $k = 2$. Then $s = 2, k = 2, t = 7$ and $s' = 1, k' = 5, t' = 5$. Substitute them back into $f(s, k, t; x)$ and $f(s', k', t'; x)$, and by simple computation, we get

$$f(s,k,t;x) = f(s',k',t';x) = -2x^8 - 2x^{10} + x^{12} - 3x^{14} + 3x^{16} - x^{18} + 2x^{20} + 2x^{22}.$$

Then graphs $H(12; 6, 1, 5)$ and $H(12; 8, 2, 2)$ are A-cospectral.

Case 9. $-2x^{t+3} = -2x^{t'+3}$ (the hypotheses are $t+1 = 2s+4$, $k > s+1$ and $t'+1 = 2s'+4$, $k' > s'+1$). Then $t = t'$ and $s = s'$, together with (3.4), we obtain that $k = k'$.

Case 10. $-4x^{t+3} = -4x^{t'+3}$ (the hypotheses are $t+1 = 2s+4$, $k = s+1$ and $t'+1 = 2s'+4$, $k' = s'+1$). Then $t = t'$ and $s = s'$, together with (3.4), we obtain that $k = k'$.

Therefore, except for the graphs $H(12; 6, 1, 5)$ and $H(12; 8, 2, 2)$, graphs $H(n; t+1, s, k)$ and $H(n; t'+1, s', k')$ are isomorphic. This completes the proof of Theorem 3.4. □

Proposition 3.5. *Graphs $H(12; 6, 1, 5)$ and $H(12; 8, 2, 2)$ are A-cospectral.*

Remark 3.1. In this section, we have shown that graphs $H(12; 6, 1, 5)$ and $H(12; 8, 2, 2)$, which both are of the form $H(n; q, n_1, n_2)$, are A-cospectral. This case is very special. Later, in Section 5, A-cospectral graphs $H(n; 2a+6, a, a+2)$ and $\Lambda(a, a, 2a+2)$ with a being a positive even number and A-cospectral graphs $H(n; 2b, b, b)$ and $\Theta(b-2, 2b-3, b-1)$ with $b \geq 4$ being a positive even number are found out in Lemmas 5.8 and 5.11, respectively. Clearly, for the respective A-cospectral mates, one is of the form $H(n; q, n_1, n_2)$, the other not. Also the same phenomena occurs in Fig. 2. We consider if there would be more graphs A-cospectral with the graph of the form $H(n; q, n_1, n_2)$. So characterizing the graph $H(n; q, n_1, n_2)$ by its A-spectrum is more complicated than the lollipop graph.

4. L-spectral characterization of graphs $H(n; q, n_1, n_2)$

From the previous section, we saw that it is very difficult to prove that graph $H(n; q, n_1, n_2)$ is determined by its A-spectrum. But, here we can prove that it is determined by its L-spectrum. Before this, we give some useful lemmas.

Lemma 4.1. *Let G be a connected unicyclic graph with n vertices and its cycle C_q. If G' is L-cospectral to G, then G' must be a connected unicyclic graph with n vertices and one cycle C_q. Moreover,*

$$\sum_{i=1}^{n} d_i(G)^3 = \sum_{i=1}^{n} d_i(G')^3.$$

Proof. By Lemma 2.2, G' is a connected graph with n vertices and n edges. So, G' is a unicyclic graph which contains a q-cycle, where q is the number of spanning tree of G' (given by the Laplacian spectrum, Lemma 2.2). As a consequence, G and G' have the same number of triangles and we can apply Lemma 2.5 and (ix) of Lemma 2.2, $\sum_{i=1}^{n} d_i(G)^3 = \sum_{i=1}^{n} d_i(G')^3$. □

Here, we use the symbol Φ to denote a forest. It is the union of components each of which is a tree. We use the symbol $p(\Phi)$ to denote the product of the numbers of vertices in the components of Φ. In [1], the following result can be found.

Lemma 4.2. *The coefficients l_i of the polynomial $P_{L(G)}(\mu)$ are given by the formula*

$$(-1)^i l_i = \sum p(\Phi) \quad (1 \leq i \leq n),$$

where the summation is over all sub-forests Φ of G which have i edges.

Theorem 4.3. *No two non-isomorphic graphs of the form $H(n; q, n_1, n_2)$ are L-cospectral.*

Proof. Suppose $H(n; q', n_1', n_2')$ is L-cospectral to $H(n; q, n_1, n_2)$. By Lemma 4.1, $q' = q$. Then, $n_1 + n_2 = n_1' + n_2'$. In the following, we use Lemma 4.2 to prove that $H(n; q, n_1', n_2')$ and $H(n; q, n_1, n_2)$ are isomorphic. We consider the coefficients l_{n-2} and l'_{n-2} of $P_{L(H(n;q,n_1,n_2))}(\mu)$ and $P_{L(H(n;q,n_1',n_2'))}(\mu)$ respectively.

For l_{n-2}, by Lemma 4.2, we get

$$(-1)^{n-2} l_{n-2} = q \sum_{i=0}^{n_1-1} (q+n_2+i)(n_1-i) + q \sum_{i=0}^{n_2-1} (q+n_1+i)(n_2-i) + \sum p(\Phi), \tag{4.1}$$

where Φ is over all sub-forests of $H(n; q, n_1, n_2)$ with $n-2$ edges obtained by deleting two edges both from C_q.

$$\begin{aligned}\sum_{i=0}^{n_1-1}(q+n_2+i)(n_1-i) &= \sum_{i=0}^{n_1-1}(q+n_2)n_1 + \sum_{i=1}^{n_1-1}(n_1-q-n_2)i - \sum_{i=0}^{n_1-1} i^2 \\ &= (q+n_2)n_1^2 + (n_1-q-n_2)\frac{n_1(n_1-1)}{2} - \frac{1}{6}n_1(n_1-1)(2n_1-1) \\ &= \frac{1}{2}qn_1^2 + \frac{1}{2}n_2n_1^2 + \frac{1}{6}n_1^3 + \frac{1}{2}n_1n_2 + \frac{1}{2}qn_1 - \frac{1}{6}n_1. \end{aligned} \tag{4.2}$$

Similarly,

$$\sum_{i=0}^{n_2-1}(q+n_1+i)(n_2-i) = \frac{1}{2}qn_2^2 + \frac{1}{2}n_1n_2^2 + \frac{1}{6}n_2^3 + \frac{1}{2}n_1n_2 + \frac{1}{2}qn_2 - \frac{1}{6}n_2. \tag{4.3}$$

Then substituting (4.2) and (4.3) into (4.1), we have

$$(-1)^{n-2}l_{n-2} = q\left((1-q)n_1n_2 + \frac{1}{2}q(n_1+n_2)^2 + \frac{1}{6}(n_1+n_2)^3 + \left(\frac{1}{2}q - \frac{1}{6}\right)(n_1+n_2)\right) + \sum p(\Phi).$$

Similarly, for l'_{n-2}, we have

$$(-1)^{n-2}l'_{n-2} = q\left((1-q)n'_1n'_2 + \frac{1}{2}q(n'_1+n'_2)^2 + \frac{1}{6}(n'_1+n'_2)^3 + \left(\frac{1}{2}q - \frac{1}{6}\right)(n'_1+n'_2)\right) + \sum p(\Phi'),$$

where Φ' is over all sub-forests of $H(n; q, n'_1, n'_2)$ with $n-2$ edges obtained by deleting two edges both from C_q.

Since $n_1 + n_2 = n'_1 + n'_2$, we have $\sum p(\Phi) = \sum p(\Phi')$. Then $l_{n-2} = l'_{n-2}$ implies that $n_1n_2 = n'_1n'_2$. Together with $n_1 + n_2 = n'_1 + n'_2$, we get that $n_1 = n'_1$ and $n_2 = n'_2$ or $n_1 = n'_2$ and $n_2 = n'_1$. Therefore, $H(n; q', n'_1, n'_2)$ and $H(n; q, n_1, n_2)$ are isomorphic. □

In the following, we will prove that graph $H(n; q, n_1, n_2)$ is determined by its L-spectrum. Before proceeding, we need to recall a few facts from the theory of nonnegative matrices; our basic references are Chapter XIII of Gantmacher [9] and [3,12]. Briefly, a matrix M is said to be *nonnegative* if $M_{ij} \geq 0$ for all i and j. A matrix is *reducible* if and only if it can be placed into block upper-triangular form by simultaneous row/column permutations. A square matrix that is not reducible is said to be *irreducible*. If M is a matrix, denote by $|M|$ the matrix obtained by replacing each entry of M by its absolute value. Denote by $\rho(|M|)$ the spectral radius of $|M|$. If M is irreducible and λ is an eigenvalue of M, then $|\lambda| \leq \rho(|M|)$, with equality if and only if $M = e^{i\phi}N|M|N^{-1}$, where $|N| = I$. For an irreducible nonnegative matrix M, $\rho(M) \leq$ the maximum row sum with equality if and only it all row sums are equal.

Now, let $L_u(H(n; q, n_1, n_2))$ be the principal submatrix of $L(H(n; q, n_1, n_2))$ formed by deleting the row and column corresponding to the largest degree vertex u. Here, $L_u(H(n; q, n_1, n_2))$ is reducible and contains negative entries. So, consider $|L_u(H(n; q, n_1, n_2))|$, which is nonnegative. Although $|L_u(H(n; q, n_1, n_2))|$ is reducible, it contains three irreducible principal submatrices. And their spectral radii are all strictly less than 4, by using the above statements. Therefore, together with eigenvalue interlacing, we have the following lemma.

Lemma 4.4. *The second largest Laplacian eigenvalue of graph $H(n; q, n_1, n_2)$ is strictly less than* 4.

Theorem 4.5. *Graph $H(n; q, n_1, n_2)$ is determined by its L-spectrum.*

Proof. Let $G = H(n; q, n_1, n_2)$. Suppose G' is L-cospectral to G. By Lemma 4.1, G' is a connected unicyclic graph with n vertices, n edges and cycle C_q. Suppose that G' has x'_j vertices of degree j, for $j = 1, 2, \ldots, \Delta$, where Δ is the largest degree of G'. By Lemma 2.6, $5 \leq \mu_1(G') = \mu_1(G) \leq 5 + \frac{2}{3}$. Then, $\Delta = d_1(G') \leq 4$. Lemmas 2.7 and 4.4 imply $d_2(G') \leq \mu_2(L(G)) < 4$, i.e., $d_2(G') \leq 3$. So, G' has at most one vertex of degree greater than 3. Therefore, (i), (ii) and (ix) of Lemma 2.2 imply the following equations:

$$\begin{cases} x'_1 + x'_2 + x'_3 + 1 = n, \\ x'_1 + 2x'_2 + 3x'_3 + \Delta = 2n, \\ x'_1 + 4x'_2 + 9x'_3 + \Delta^2 = 2 + 4(n-3) + 4^2. \end{cases} \tag{4.4}$$

By solving the Eqs. (4.4), we have

$$x'_1 = \frac{5}{2}\Delta - \frac{1}{2}\Delta^2, \qquad x'_2 = n - 3 - 4\Delta + \Delta^2, \qquad x'_3 = \frac{3}{2}\Delta - \frac{1}{2}\Delta^2 + 2. \tag{4.5}$$

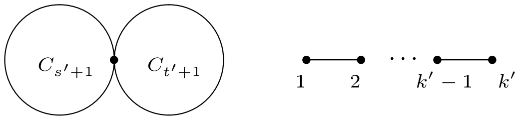

Fig. 4. Graph $\Gamma_2 = G_{s'+1,t'+1} \bigcup P_{k'}$.

Lemma 4.1 implies that

$$\Delta^3 + x'_3 \cdot 3^3 + x'_2 \cdot 2^3 + x'_1 \cdot 1^3 = 4^3 + (n-3) \cdot 2^3 + 2 \cdot 1^3. \tag{4.6}$$

Substitute (4.5) into (4.6), we have

$$\Delta^3 + \left(\frac{3}{2}\Delta - \frac{1}{2}\Delta^2 + 2\right) \cdot 3^3 + \left(n - 3 - 4\Delta + \Delta^2\right) \cdot 2^3 + \left(\frac{5}{2}\Delta - \frac{1}{2}\Delta^2\right) \cdot 1^3 = 4^3 + (n-3) \cdot 2^3 + 2 \cdot 1^3. \tag{4.7}$$

By simplifying (4.7), we get $(\Delta - 4)(\Delta^2 - 2\Delta + 3) = 0$. Then $\Delta = 4$, since Δ is a positive integer. Then by (4.5), we have $x'_3 = 0$, $x'_2 = n - 3$, $x'_1 = 2$. So, G' is the graph $H(n; q, n'_1, n'_2)$ (say). By Theorem 4.3, $H(n; q, n_1, n_2)$ is isomorphic to $H(n; q, n'_1, n'_2)$. □

For a graph, its Laplacian eigenvalues determine the eigenvalues of its complement [13], so the complements of all the graphs $H(n; q, n_1, n_2)$ are determined by their L-spectra.

5. Q-spectral characterization of graphs $H(n; q, n_1, n_2)$

Let $G_{r',s'}$ be the graph consisting of two cycles $C_{r'}$ and $C_{s'}$ with just one vertex in common [25]. $G \bigcup H$ stands for the disjoint union of graphs G and H. Let $\Gamma_2 = G_{s'+1,t'+1} \bigcup P_{k'}$ with $n' = s' + t' + k' + 1$ vertices, shown in Fig. 4. Clearly, if $n' = n$, then graphs $\Gamma_1 = H(n; t+1, s, k)$ and $\Gamma_2 = G_{s'+1,t'+1} \bigcup P_{k'}$ have the same degree sequence. In the following, we prove that Γ_1 and Γ_2 cannot be A-cospectral with each other.

First, we calculate the characteristic polynomial of Γ_2 by using the analogous method to Γ_1. By Lemma 2.1, with u being the vertex of degree 4 in Γ_2, we can compute the characteristic polynomial of Γ_2 in terms of the characteristic polynomials of paths as follows

$$P_{A(\Gamma_2)}(\lambda) = p_{k'}(\lambda p_{s'} p_{t'} - 2p_{s'} p_{t'-1} - 2p_{s'-1} p_{t'} - 2p_{s'} - 2p_{t'}). \tag{5.1}$$

Substituting (3.2) in (5.1), by using Maple, we can obtain

$$x^{n'}(x^2-1)^3 P_{A(\Gamma_2)}(\lambda) + 1 - 3x^2 - x^{2n'+6} + 3x^{2n'+4} = g(s', t', k'; x), \tag{5.2}$$

where $n' = s' + t' + k' + 1$ and

$$\begin{aligned} g(s', t', k'; x) = {} & 2x^{t'+1} + 2x^{s'+1} + x^{2k'+2} - 2x^{t'+3} - 2x^{s'+3} - x^{2s'+2} - x^{2+2t'} - x^{4+2t'} - x^{4+2s'} \\ & - 3x^{2k'+4} - 2x^{2k'+3+t'} + 2x^{2k'+5+t'} - 2x^{t'+3+2s'} + 2x^{t'+5+2s'} + 2x^{2k'+5+s'} - 2x^{s'+3+2t'} \\ & + 2x^{s'+5+2t'} - x^{6+2s'+2t'} - 2x^{2k'+3+s'} + x^{2k'+6+2s'} + x^{2k'+4+2s'} + x^{2k'+4+2t'} + x^{2k'+6+2t'} \\ & + 3x^{4+2s'+2t'} - 2x^{2k'+7+s'+2t'} + 2x^{2k'+5+s'+2t'} + 2x^{2k'+5+t'+2s'} - 2x^{2k'+7+t'+2s'}. \end{aligned}$$

The following proposition follows from (5.1) and $p_r(2) = r + 1$.

Proposition 5.1. $P_{A(\Gamma_2)}(2) = -2k's't' - 2k's' - 2k't' - 2s't' - 2(k' + s' + t' + 1)$.

Lemma 5.2. *There is no graph $\Gamma_1 = H(n; t+1, s, k)$ being A-cospectral with $\Gamma_2 = G_{s'+1,t'+1} \bigcup P_{k'}$.*

Proof. Suppose that $\Gamma_1 = H(n; t+1, s, k)$ and $\Gamma_2 = G_{s'+1,t'+1} \bigcup P_{k'}$ are A-cospectral, then they have the same number of vertices, that is,

$$t + s + k + 1 = s' + t' + k' + 1 = n = n'. \tag{5.3}$$

By (3.3) and (5.2), we get

$$f(s, k, t; x) = g(s', t', k'; x). \tag{5.4}$$

Without loss of generality, we can assume $s \leq k$ and $t' \leq s'$. Now enumerate the different possibilities for the monomial of smallest degree of $g(s', t', k'; x)$ are

- $2x^{t'+1}$ if $t'+1 < 2k'+2$ and $t' \neq s'$,
- $4x^{t'+1}$ if $t'+1 < 2k'+2$ and $t' = s'$,
- $3x^{t'+1}$ if $t'+1 = 2k'+2$ and $t' \neq s'$,
- $5x^{t'+1}$ if $t'+1 = 2k'+2$ and $t' = s'$,
- $x^{2k'+2}$ if $t'+1 > 2k'+2$.

And the different possibilities for the monomial of smallest degree of $f(s, k, t; x)$ are the same as (3.7). Therefore, by (5.4), we only need to consider the case $2x^{t+1} = 2x^{t'+1}$. Then $t = t'$. We define

$$f'(s, k, t; x) = f(s, k, t; x) - 2x^{t+1} + 2x^{t+3} + x^{2t+2} + x^{2t+4}$$

and

$$g'(s', t', k'; x) = g(s', t', k'; x) - 2x^{t'+1} + 2x^{t'+3} + x^{2t'+2} + x^{2t'+4}.$$

For $f'(s, k, t; x)$, the different possibilities for the monomial of smallest degree are

- $-2x^{2s+4}$ if $s < k$,
- $-4x^{2s+4}$ if $s = k$.

For $g'(s', t', k'; x)$, the different possibilities for the monomial of smallest degree are

- $2x^{s'+1}$ if $s'+1 < 2k'+2$,
- $x^{2k'+2}$ if $s'+1 > 2k'+2$,
- $3x^{s'+1}$ if $s'+1 = 2k'+2$.

Clearly, there is no case such that the coefficients are the same, a contradiction. This completes the proof of Lemma 5.2. □

For a graph G, let $\mathcal{L}(G)$ be the line graph of G, and $\mathcal{S}(G)$ be the subdivision graph of G. Recall that a subdivision graph $\mathcal{S}(G)$ is a graph obtained from G by replacing each edge of G by a path of length two [5]. Vertices of $\mathcal{L}(G)$ are in one-to-one correspondence with edges of G, and two vertices in $\mathcal{L}(G)$ are adjacent if and only if the corresponding edges of G are adjacent [5]. Note that the Q-spectrum of a graph can be exactly expressed by the A-spectrum of its line and subdivision graphs [7,4,8], and the following results can be found in [7,4,25].

Lemma 5.3. *If two graphs G and H are Q-cospectral, then their line graphs $\mathcal{L}(G)$ and $\mathcal{L}(H)$ are A-cospectral.*

Lemma 5.4. *Let G be a graph with n vertices, and $\mathcal{S}(G)$ its subdivision graph. Then graphs G and H are Q-cospectral if and only if $\mathcal{S}(G)$ and $\mathcal{S}(H)$ are A-cospectral.*

Since the subdivision graph of $H(n; q, n_1, n_2)$ is $H(2n; 2q, 2n_1, 2n_2)$, Lemma 5.4 and Theorem 3.4 imply the following result.

Lemma 5.5. *No two non-isomorphic graphs of the form $H(n; q, n_1, n_2)$ are Q-cospectral.*

For the sake of simplicity, let $G_1 = H(n, c+6, a, b)$ with c being nonnegative even numbers and a, b positive even numbers. Using Lemma 2.1, with u being the vertex of degree 4 in G_1, we can compute the characteristic polynomial of G_1 in terms of the characteristic polynomials of paths as follows

$$P_{A(G_1)}(\lambda) = \lambda p_a p_b p_{c+5} - p_{c+5} p_{a-1} p_b - p_{c+5} p_a p_{b-1} - 2p_{c+4} p_a p_b - 2p_a p_b. \tag{5.5}$$

Substituting (3.2) in (5.5), by using Maple, we can obtain

$$x^n(x^2-1)^3 P_{A(G_1)}(\lambda) + 1 - 3x^2 - x^{2n+6} + 3x^{2n+4} = f_1(a, b, c; x), \tag{5.6}$$

where $n = a + b + c + 6$ and

$$\begin{aligned} f_1(a, b, c; x) = {} & 2x^{c+6} - 2x^{2b+4} - 2x^{2a+4} - 2x^{c+8} - x^{2c+12} - x^{2c+14} - 2x^{c+8+2a} + 2x^{c+10+2a} - 2x^{c+8+2b} \\ & + 2x^{c+10+2b} + x^{2a+4+2b} + x^{2a+6+2b} + 2x^{2a+14+2c} + 2x^{14+2b+2c} + 2x^{c+10+2a+2b} - 2x^{c+12+2a+2b}. \end{aligned}$$

The next lemma follows from (5.5) and $p_r(2) = r + 1$.

Lemma 5.6. $P_{A(G_1)}(2) = -2abc - 12ab - ac - bc - 6a - 6b$.

Let $G_2 = \Lambda(a', b', c')$ be the graph shown in Fig. 5 with a', b' and c' being positive even numbers. Clearly, it is the subdivision graph of G_2' with $d(z, y) = \frac{a'}{2}$, $d(u, v) = \frac{b'}{2}$ and $d(w, x) = \frac{c'}{2}$ (shown in Fig. 10). Then, by Lemma 2.1, with u being the vertex of degree 3 in G_2, we can compute the characteristic polynomial of G_2 in terms of the characteristic

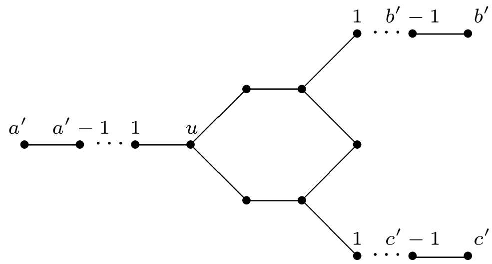

Fig. 5. Graph $G_2 = \Lambda(a', b', c')$.

polynomials of paths recursively as follows

$$\begin{aligned} P_{A(G_2)}(\lambda) &= \lambda p_{a'}(\lambda p_{b'+2}p_{c'+2} - p_1p_{b'}p_{c'+2} - p_1p_{b'+2}p_{c'}) - p_{a'-1}(\lambda p_{b'+2}p_{c'+2} - p_1p_{b'}p_{c'+2} - p_1p_{b'+2}p_{c'}) \\ &\quad - p_{a'}(\lambda p_{b'+1}p_{c'+2} - p_{b'}p_{c'+2} - p_1p_{b'+1}p_{c'}) - p_{a'}(\lambda p_{b'+2}p_{c'+1} - p_{b'+2}p_{c'} - p_1p_{b'}p_{c'+1}) - 2p_{a'}p_{b'}p_{c'} \\ &= \lambda^2 p_{a'}p_{b'+2}p_{c'+2} - \lambda p_1p_{a'}p_{b'}p_{c'+2} - \lambda p_1p_{a'}p_{b'+2}p_{c'} - \lambda p_{a'-1}p_{b'+2}p_{c'+2} + p_1p_{a'-1}p_{b'}p_{c'+2} \\ &\quad + p_1p_{a'-1}p_{b'+2}p_{c'} - \lambda p_{a'}p_{b'+1}p_{c'+2} + p_{a'}p_{b'}p_{c'+2} + p_1p_{a'}p_{b'+1}p_{c'} - \lambda p_{a'}p_{b'+2}p_{c'+1} \\ &\quad + p_{a'}p_{b'+2}p_{c'} + p_1p_{a'}p_{b'}p_{c'+1} - 2p_{a'}p_{b'}p_{c'}. \end{aligned} \tag{5.7}$$

Substituting (3.2) in (5.7), by using Maple, we can obtain

$$x^{n'}(x^2-1)^3 P_{A(G_2)}(\lambda) + 1 - 3x^2 - x^{2n'+6} + 3x^{2n'+4} = f_2(a', b', c'; x), \tag{5.8}$$

where $n' = a' + b' + c' + 6$ and

$$\begin{aligned} f_2(a', b', c'; x) &= -x^{2a'+4} - x^{2b'+4} - x^{2c'+4} + x^{2a'+6} + x^{2b'+6} + x^{2c'+6} - 2x^{2a'+8} - 2x^{2b'+8} - 2x^{2c'+8} \\ &\quad + 2x^{2a'+2b'+10} + 2x^{2a'+2c'+10} + 2x^{2b'+2c'+10} - x^{2a'+2b'+12} - x^{2a'+2c'+12} \\ &\quad - x^{2b'+2c'+12} + x^{2a'+2b'+14} + x^{2a'+2c'+14} + x^{2b'+2c'+14}. \end{aligned}$$

By (5.7) and $p_r(2) = r + 1$, we have the following lemma.

Lemma 5.7. $P_{A(G_2)}(2) = -2a'b'c' - 4a'b' - 4a'c' - 4b'c' - 6a' - 6b' - 6c'$.

Lemma 5.8. *Except for the A-cospectral graphs $H(n; 2a+6, a, a+2)$ and $\Lambda(a, a, 2a+2)$, there is no graph $G_1 = H(n; c+6, a, b)$ being A-cospectral with $G_2 = \Lambda(a', b', c')$.*

Proof. Suppose that G_1 and G_2 are A-cospectral, then they have the same number of vertices, that is,

$$a + b + c + 6 = a' + b' + c' + 6 = n = n'. \tag{5.9}$$

By Lemmas 5.6 and 5.7, we have

$$-2abc - 12ab - ac - bc - 6a - 6b = -2a'b'c' - 4a'b' - 4a'c' - 4b'c' - 6a' - 6b' - 6c'. \tag{5.10}$$

Then Eqs. (5.6) and (5.8) imply that

$$f_1(a, b, c; x) = f_2(a', b', c'; x). \tag{5.11}$$

Without loss of generality, we can assume $a \le b$. Now enumerate the different possibilities for the monomial of smallest degree of $f_1(a, b, c; x)$:

(5.12)

- $2x^{c+6}$ if $c + 6 < 2a + 4$,
- $-2x^{2a+4}$ if $c + 6 > 2a + 4$ and $a \ne b$,
- $-4x^{2a+4}$ if $c + 6 > 2a + 4$ and $a = b$,
- $-2x^{2b+4}$ if $c + 6 = 2a + 4$ and $c + 8 > 2b + 4$ (that is $a = b$),
- $-2x^{c+8}$ if $c + 6 = 2a + 4$ and $c + 8 < 2b + 4$ (that is $b > a + 1$),
- $-4x^{c+8}$ if $c + 6 = 2a + 4$ and $c + 8 = 2b + 4$ (that is $b = a + 1$).

Similarly, without loss of generality, we can assume $a' \le b' \le c'$. Now enumerate the different possibilities for the monomial of smallest degree of $f_2(a', b', c'; x)$:

- $-x^{2a'+4}$ if $a' < b'$,
- $-2x^{2a'+4}$ if $a' = b' < c'$,
- $-3x^{2a+4}$ if $a' = b' = c'$.

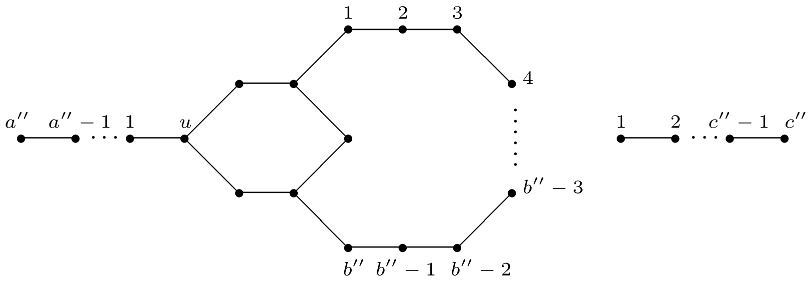

Fig. 6. Graph $G_3 = \Theta(a'', b'', c'')$.

Then by (5.11), it suffices to consider the following cases:

Case 1. $-2x^{2a+4} = -2x^{2a'+4}$ (the hypotheses are $c + 6 > 2a + 4$, $a \neq b$ and $a' = b' < c'$). Then $a' = b' = a$. By (5.9), we have $c' = b + c - a$. Substitute $a' = a$ and $b' = a$ into $f_2(a', b', c'; x)$, then

$$f_2(a', b', c'; x) = -2x^{2a+4} - x^{2c'+4} + 2x^{2a+6} + x^{2c'+6} - 4x^{2a+8} - 2x^{2c'+8} + 2x^{4a+10} + 4x^{2a+2c'+10} - x^{4a+12} - 2x^{2a+2c'+12} + x^{4a+14} + 2x^{2a+2c'+14}.$$

Since $c' > a$ and c', a are both positive even numbers, clearly $c' > a + 1$. Thus $2c' + 4 > 2a + 6$. Then the second smallest term of $f_2(a', b', c'; x)$ is $2x^{2a+6}$. And the different possibilities for the second smallest term of $f_1(a, b, c; x)$ are

$$\begin{aligned} &\bullet\ -2x^{2b+4} \quad \text{if } 2b+4 < c+6,\\ &\bullet\ 2x^{c+6} \quad \text{if } 2b+4 > c+6,\\ &\bullet\ -2x^{c+8} \quad \text{if } 2b+4 = c+6. \end{aligned} \tag{5.13}$$

So (5.11) implies that $2x^{c+6} = 2x^{2a+6}$, then $c = 2a$ and $c' = b + c - a = a + b$. Substituting $a, b, c = 2a, a' = a, b' = a$ and $c' = a + b$ into (5.10), we have $b = a + 2$. Clearly,

$$\begin{aligned} f_1(a, b, c; x) &= f_2(a', b', c'; x)\\ &= -2x^{2a+4} + 2x^{2a+6} - 4x^{2a+8} - x^{4a+8} + 3x^{4a+10} - 3x^{4a+12} + x^{4a+14} + 4x^{6a+14} - 2x^{6a+16} + 2x^{6a+18}. \end{aligned}$$

Thus, graphs $H(n; 2a + 6, a, a + 2)$ and $\Lambda(a, a, 2a + 2)$ are A-cospectral.

Case 2. $-2x^{2b+4} = -2x^{2a'+4}$ (the hypotheses are $c + 6 = 2a + 4$, $a = b$ and $a' = b' < c'$). Then $a = b = a' = b'$ and $c = 2a - 2$. By (5.9), we have $c' = c = 2a - 2$. Substituting $a, b = a, c = 2a - 2, a' = a, b' = a$ and $c' = 2a - 2$ into (5.10), we have $a^2 - 3 = 0$, which is a contradiction to the fact that a is a positive even number.

Case 3. $-2x^{c+8} = -2x^{2a'+4}$ (the hypotheses are $c + 6 = 2a + 4$, $b > a + 1$ and $a' = b' < c'$). Then $a' = b' = a + 1$, which are contradictions to the fact that a, a' and b' are positive even numbers.

Therefore, except for the A-cospectral graphs $H(n; 2a + 6, a, a + 2)$ and $\Lambda(a, a, 2a + 2)$, there is no graph $G_1 = H(n; c + 6, a, b)$ being A-cospectral with $G_2 = \Lambda(a', b', c')$. This completes the proof of Lemma 5.8. □

Lemmas 5.4 and 5.8 imply the following result, since $H(4a+8; 2a+6, a, a+2)$ is the subdivision graph of $H(2a+4; a+3, \frac{a}{2}, \frac{a}{2}+1)$ and $G_2 = \Lambda(a, a, 2a+2)$ is the subdivision graph of G_2' with $d(z, y) = \frac{a}{2}$, $d(u, v) = \frac{a}{2}$ and $d(w, x) = a + 1$ (shown in Fig. 10).

Lemma 5.9. *Except for the graph $H(2a + 4; a + 3, \frac{a}{2}, \frac{a}{2} + 1)$ with a being a positive even number, there is no graph of the form $H(n; q, n_1, n_2)$ being Q-cospectral with graph G_2'. And graph $H(2a + 4; a + 3, \frac{a}{2}, \frac{a}{2} + 1)$ is Q-cospectral with graph G_2' with $d(z, y) = \frac{a}{2}$, $d(u, v) = \frac{a}{2}$ and $d(w, x) = a + 1$.*

Let $G_3 = \Theta(a'', b'', c'')$ be the graph shown in Fig. 6 with $b'' \geq 3$ and $c'' \geq 3$ being odd numbers, and a'' a positive even number. Clearly, it is the subdivision graph of G_4' with $q' = \frac{b''+3}{2}$, $d(u, v) = \frac{c''-1}{2}$ and $d(x, y) = \frac{a''}{2}$ (shown in Fig. 10). Then, by Lemma 2.1, with u being the vertex of degree 3 in G_3, we can compute the characteristic polynomial of G_3 in terms of the characteristic polynomials of paths recursively as follows

$$\begin{aligned} P_{A(G_3)}(\lambda) &= p_{c''}(\lambda p_{a''}(\lambda p_{b''+4} - 2p_1 p_{b''+2} - 2p_1^2) - p_{a''-1}(\lambda p_{b''+4} - 2p_1 p_{b''+2} - 2p_1^2)\\ &\quad - 2p_{a''}(\lambda p_{b''+3} - p_{b''+2} - p_1 p_{b''+1} - 2p_1) - 2p_{a''} p_{b''} - 2p_1 p_{a''})\\ &= \lambda^2 p_{a''} p_{b''+4} p_{c''} - 2\lambda p_1 p_{a''} p_{b''+2} p_{c''} - 2\lambda p_1^2 p_{a''} p_{c''} - \lambda p_{a''-1} p_{b''+4} p_{c''}\\ &\quad + 2p_1 p_{a''-1} p_{b''+2} p_{c''} + 2p_1^2 p_{a''-1} p_{c''} - 2\lambda p_{a''} p_{b''+3} p_{c''} + 2p_{a''} p_{b''+2} p_{c''}\\ &\quad + 2p_1 p_{a''} p_{b''+1} p_{c''} + 2p_1 p_{a''} p_{c''} - 2p_{a''} p_{b''} p_{c''}. \end{aligned} \tag{5.14}$$

Substituting (3.2) in (5.14), by using Maple, we can obtain

$$x^{n''}(x^2-1)^3 P_{A(G_3)}(\lambda) + 1 - 3x^2 - x^{2n''+6} + 3x^{2n''+4} = f_3(a'', b'', c''; x), \tag{5.15}$$

where $n'' = a'' + b'' + c'' + 6$ and

$$\begin{aligned} f_3(a'', b'', c''; x) = {} & 2x^{b''+3} - 2x^{b''+7} - 2x^{2b''+8} + x^{2b''+10} - x^{2b''+12} - x^{2a''+4} + x^{2a''+6} - 2x^{2a''+8} + x^{2c''+2} \\ & - 3x^{2c''+4} - 2x^{b''+2a''+9} + 2x^{b''+2a''+13} - 2x^{b''+2c''+5} + 2x^{b''+2c''+9} + 3x^{2a''+2b''+14} \\ & - x^{2a''+2b''+16} + x^{2a''+2c''+6} - x^{2a''+2c''+8} + 2x^{2a''+2c''+10} + 2x^{2b''+2c''+10} \\ & - x^{2b''+2c''+12} + x^{2b''+2c''+14} + 2x^{2a''+b''+2c''+11} - 2x^{2a''+b''+2c''+15}. \end{aligned}$$

By (5.14) and $p_r(2) = r + 1$, we have the following lemma.

Lemma 5.10. $P_{A(G_3)}(2) = -2a''b''c'' - 2a''b'' - 10a''c'' - 4b''c'' - 10a'' - 4b'' - 20c'' - 20$.

Lemma 5.11. *Except for the A-cospectral graphs $H(n; 2b, b, b)$ and $\Theta(b-2, 2b-3, b-1)$, there is no graph $G_1 = H(n; c+6, a, b)$ being A-cospectral with $G_3 = \Theta(a'', b'', c'')$.*

Proof. Suppose that G_1 and G_3 are A-cospectral, then they have the same number of vertices, that is,

$$a + b + c + 6 = a'' + b'' + c'' + 6 = n = n''. \tag{5.16}$$

By Lemmas 5.6 and 5.10, we have

$$-2abc - 12ab - ac - bc - 6a - 6b = -2a''b''c'' - 2a''b'' - 10a''c'' - 4b''c'' - 10a'' - 4b'' - 20c'' - 20. \tag{5.17}$$

Then Eqs. (5.6) and (5.15) imply that

$$f_1(a, b, c; x) = f_3(a'', b'', c''; x). \tag{5.18}$$

The different possibilities for the monomial of smallest degree of $f_1(a, b, c; x)$ are the same as (5.12) in Lemma 5.8. Now enumerate the different possibilities for the monomial of smallest degree of $f_3(a'', b'', c''; x)$:

- $2x^{b''+3}$ if $b''+3 < 2a''+4$ and $b''+3 < 2c''+2$,
- $x^{b''+3}$ if $b''+3 = 2a''+4 < 2c''+2$,
- $3x^{b''+3}$ if $b''+3 = 2c''+2 < 2a''+4$,
- $-x^{2a''+4}$ if $2a''+4 < b''+3$ and $2a''+4 < 2c''+2$,
- $2x^{b''+3}$ if $2a''+4 = 2c''+2 < b''+3 < 2c''+4$,
- $-2x^{2c''+4}$ if $2a''+4 = 2c''+2 < b''+3$ and $2c''+4 < b''+3$,
- $-2x^{2a''+8}$ if $2a''+4 = 2c''+2 < b''+3 = 2c''+4$,
- $x^{2c''+2}$ if $2c''+2 < b''+3$ and $2c''+2 < 2a''+4$,
- $2x^{b''+3}$ if $2a''+4 = b''+3 = 2c''+2$.

Then by (5.18), it suffices to consider the following cases:

Case 1. $2x^{c+6} = 2x^{b''+3}$ (the hypotheses are $c+6 < 2a+4$ and $b''+3 < 2a''+4$, $b''+3 < 2c''+2$). Then $b'' = c+3$. Now the different possibilities for the monomial of second smallest degree of $f_1(a, b, c; x)$ are

$$\begin{aligned} & \bullet\ -2x^{2a+4} \quad \text{if } 2a+4 < c+8 \text{ and } a \neq b, \\ & \bullet\ -4x^{2a+4} \quad \text{if } 2a+4 < c+8 \text{ and } a = b, \\ & \bullet\ -2x^{c+8} \quad \text{if } 2a+4 > c+8, \\ & \bullet\ -4x^{2a+4} \quad \text{if } 2a+4 = c+8 \text{ and } a \neq b, \\ & \bullet\ -6x^{2a+4} \quad \text{if } 2a+4 = c+8 \text{ and } a = b. \end{aligned} \tag{5.19}$$

And the different possibilities for the monomial of second smallest degree of $f_3(a'', b'', c''; x)$ are

- $-2x^{b''+7}$ if $b''+7 < 2a''+4$ and $b''+7 < 2c''+2$,
- $-3x^{b''+7}$ if $b''+7 = 2a''+4 < 2c''+2$,
- $-x^{b''+7}$ if $b''+7 = 2c''+2 < 2a''+4$,
- $-x^{2a''+4}$ if $2a''+4 < b''+7$ and $2a''+4 < 2c''+2$,
- $-2x^{b''+7}$ if $2a''+4 = 2c''+2 < b''+7 < 2c''+4$,

- $-2x^{2c''+4}$ if $2a''+4=2c''+2<b''+7$ and $2c''+4<b''+7$,
- $-4x^{2c''+4}$ if $2a''+4=2c''+2<b''+7=2c''+4$,
- $x^{2c''+2}$ if $2c''+2<b''+7$ and $2c''+2<2a''+4$,
- $-2x^{b''+7}$ if $2a''+4=b''+7=2c''+2$.

Then by (5.18), we consider the following subcases:

Case 1.1. $-2x^{2a+4}=-2x^{b''+7}$ (the hypotheses are $2a+4<c+8$, $a\neq b$ and $b''+7<2a''+4$, $b''+7<2c''+2$). Then $2a+4=b''+7=c+10>c+8$, a contradiction to $2a+4<c+8$.

Case 1.2. $-2x^{2a+4}=-2x^{b''+7}$ (the hypotheses are $2a+4<c+8$, $a\neq b$ and $2a''+4=2c''+2<b''+7<2c''+4$). Then $2a+4=b''+7=c+10>c+8$, a contradiction to $2a+4<c+8$.

Case 1.3. $-2x^{2a+4}=-2x^{2c''+4}$ (the hypotheses are $2a+4<c+8$, $a\neq b$ and $2a''+4=2c''+2<b''+7$, $2c''+4<b''+7$). Then $a=c''$, which is a contradiction to the fact that a is a positive even number and c'' is a positive odd number.

Case 1.4. $-2x^{2a+4}=-2x^{b''+7}$ (the hypotheses are $2a+4<c+8$, $a\neq b$ and $2a''+4=b''+7=2c''+2$). Then $2a+4=b''+7=c+10>c+8$, a contradiction to $2a+4<c+8$.

Case 1.5. $-4x^{2a+4}=-4x^{2c''+4}$ (the hypotheses are $2a+4<c+8$, $a=b$ and $2a''+4=2c''+2<b''+7=2c''+4$). Then $a=c''$, which is a contradiction to the fact that a is a positive even number and c'' is a positive odd number.

Case 1.6. $-2x^{c+8}=-2x^{b''+7}$ (the hypotheses are $c+8<2a+4$ and $b''+7<2a''+4$, $b''+7<2c''+2$). Then $b''=c+1$, which is a contradiction to $b''=c+3$.

Case 1.7. $-2x^{c+8}=-2x^{b''+7}$ (the hypotheses are $c+8<2a+4$ and $2a''+4=2c''+2<b''+7<2c''+4$). Then $b''=c+1$, which is a contradiction to $b''=c+3$.

Case 1.8. $-2x^{c+8}=-2x^{2c''+4}$ (the hypotheses are $c+8<2a+4$ and $2a''+4=2c''+2<b''+7$, $2c''+4<b''+7$). Then $c=2c''-4=2a''-2$, so $b''=c+3=2a''+1$. Since $b''+3$, $2a''+4$ and $b''+7$ are positive even numbers, $b''+3<2a''+4<b''+7$ implies that $2a''+4=b''+5$, that is $b''=2a''-1$, which is a contradiction to $b''=2a''+1$.

Case 1.9. $-2x^{c+8}=-2x^{b''+7}$ (the hypotheses are $c+8<2a+4$ and $2a''+4=b''+7=2c''+2$). Then $b''=c+1$, which is a contradiction to $b''=c+3$.

Case 1.10. $-4x^{c+8}=-4x^{2c''+4}$ (the hypotheses are $2a+4=c+8$, $a\neq b$ and $2a''+4=2c''+2<b''+7=2c''+4$). Then $c+8=2c''+4=b''+7$ implies that $b''=c+1$, which is a contradiction to $b''=c+3$.

Case 2. $2x^{c+6}=2x^{b''+3}$ (the hypotheses are $c+6<2a+4$ and $2a''+4=2c''+2<b''+3<2c''+4$). Indeed, $2c''+2$, $b''+3$ and $2c''+4$ are positive even numbers, there does not exist $b''+3$ such that $2c''+2<b''+3<2c''+4$, so the hypotheses are not valid.

Case 3. $2x^{c+6}=2x^{b''+3}$ (the hypotheses are $c+6<2a+4$ and $2a''+4=b''+3=2c''+2$). Then $b''=c+3$. Now the monomial of second smallest degree of $f_3(a'',b'',c'';x)$ is $-2x^{2c''+4}$, and the different possibilities for the monomial of second smallest degree of $f_1(a,b,c;x)$ are the same as (5.19). Then by (5.18), it suffices to consider the following subcases:

Case 3.1. $-2x^{2a+4}=-2x^{2c''+4}$ (the hypotheses are $2a+4<c+8$ and $a\neq b$). Then $2a+4=2c''+4=b''+5=c+8$, which is a contradiction to $2a+4<c+8$.

Case 3.2. $-2x^{c+8}=-2x^{2c''+4}$ (the hypothesis is $2a+4>c+8$). Substituting $2a''=b''-1$ and $2c''=b''+1$ into $f_3(a'',b'',c'';x)$, we have

$$f_3(a'',b'',c'';x)=2x^{b''+3}-2x^{b''+5}-4x^{b''+7}-x^{2b''+6}-5x^{2b''+8}+5x^{2b''+10}+x^{2b''+12}+4x^{3b''+11}+2x^{3b''+13}-2x^{3b''+15}.$$

Clearly, the monomial of third smallest degree of $f_3(a'',b'',c'';x)$ is $-4x^{b''+7}$. By substituting $c=b''-3$ into $f_1(a,b,c;x)$, we have

$$f_1(a,b,c;x)=2x^{b''+3}-2x^{b''+5}-2x^{2b+4}-2x^{2a+4}-x^{2b''+6}-x^{2b''+8}-2x^{b''+5+2a}+2x^{b''+7+2a}-2x^{b''+5+2b}+2x^{b''+7+2b}+x^{2a+4+2b}+x^{2a+6+2b}+2x^{2a+8+2b''}+2x^{8+2b+2b''}+2x^{b''+7+2a+2b}-2x^{b''+9+2a+2b}.$$

Now the different possibilities for the monomial of third smallest degree of $f_1(a,b,c;x)$ are

- $-2x^{2a+4}$ if $2a+4<2b''+6$ and $a\neq b$,
- $-4x^{2a+4}$ if $2a+4<2b''+6$ and $a=b$,
- $-x^{2b''+6}$ if $2a+4>2b''+6$,
- $-3x^{2a+4}$ if $2a+4=2b''+6$ and $a\neq b$,
- $-5x^{2a+4}$ if $2a+4=2b''+6$ and $a=b$.

Then by (5.18), we have $-4x^{b''+7} = -4x^{2a+4}$ (the hypotheses are $2a+4 < 2b''+6$ and $a = b$). Then $a = b = \frac{b''+3}{2}$. Clearly,

$$\begin{aligned} f_1(a, b, c; x) &= f_3(a'', b'', c''; x) \\ &= 2x^{b''+3} - 2x^{b''+5} - 4x^{b''+7} - x^{2b''+6} - 5x^{2b''+8} + 5x^{2b''+10} \\ &\quad + x^{2b''+12} + 4x^{3b''+11} + 2x^{3b''+13} - 2x^{3b''+15}. \end{aligned}$$

Now we have $a = b, b'' = 2b-3, c = 2b-6, a'' = b-2$ and $c'' = b-1$. Thus, graphs $H(n; 2b, b, b)$ and $\Theta(b-2, 2b-3, b-1)$ are A-cospectral, where $b \geq 4$ is a positive even number.

Case 4. $-2x^{2a+4} = -2x^{2c''+4}$ (the hypotheses are $c+6 > 2a+4, a \neq b$ and $2a''+4 = 2c''+2 < b''+3, 2c''+4 < b''+3$). Then $c'' = a$, which is a contradiction to the fact that c'' is a positive odd number and a is a nonnegative even number.

Case 5. $-2x^{2a+4} = -2x^{2a''+8}$ (the hypotheses are $c+6 > 2a+4, a \neq b$ and $2a''+4 = 2c''+2 < b''+3 = 2c''+4$). Then $a'' = a-2, b'' = 2a-1$ and $c'' = a-1$. Substituting $a'' = a-2, b'' = 2a-1$ and $c'' = a-1$ into $f_3(a'', b'', c''; x)$, we get

$$f_3(a'', b'', c''; x) = -2x^{2a+4} - 2x^{2a+6} + x^{4a} - 3x^{4a+2} + 3x^{4a+8} - x^{4a+10} + 2x^{6a+4} + 2x^{6a+6}.$$

Now the monomial of second smallest degree of $f_3(a'', b'', c''; x)$ is $-2x^{2a+6}$. The different possibilities for the monomial of second smallest degree of $f_1(a, b, c; x)$ is the same as (5.13) in Lemma 5.8. Then by (5.18), consider the following subcases:

Case 5.1. $-2x^{2a+6} = -2x^{2b+4}$ (the hypothesis is $2b+4 < c+6$). Then $b = a+1$, which is a contradiction to the fact that a and b are positive even numbers.

Case 5.2. $-2x^{2a+6} = -2x^{c+8}$ (the hypothesis is $2b+4 = c+6$). Then $a = b$, a contradiction to $a \neq b$.

Case 6. $-2x^{2b+4} = -2x^{2c''+4}$ (the hypotheses are $c+6 = 2a+4, a = b$ and $2a''+4 = 2c''+2 < b''+3, 2c''+4 < b''+3$). Then $c'' = b$, which is a contradiction to the fact that c'' is a positive odd number and b is a positive even number.

Case 7. $-2x^{2b+4} = -2x^{2a''+8}$ (the hypotheses are $c+6 = 2a+4, a = b$ and $2a''+4 = 2c''+2 < b''+3 = 2c''+4$). Then we have $b = a, c = 2a-2, a'' = a-2, c'' = a-1, b'' = 2a-1$. But $a+b+c = 4a-2 \neq a''+b''+c'' = 4a-4$, a contradiction to (5.16).

Case 8. $-2x^{c+8} = -2x^{2c''+4}$ (the hypotheses are $c+6 = 2a+4, b > a+1$ and $2a''+4 = 2c''+2 < b''+3, 2c''+4 < b''+3$). Then we have $c = 2a-2, c'' = a+1$ and $a'' = a$. By (5.16), we have $b'' = a+b-3$. Substituting $c = 2a-2, c'' = a+1$, $a'' = a$ and $b'' = a+b-3$ into (5.17), we have $b = a+4+\frac{8}{a-1}$. Then $a = 2$ and $b = 14$, since a and b are both positive even numbers. So we have $a = 2, b = 14, c = 2$ and $a'' = 2, b'' = 13, c'' = 3$. Substitute them back into $f_1(a, b, c; x)$ and $f_3(a'', b'', c''; x)$ respectively, and by simple computation, we get $f_1(a, b, c; x) \neq f_3(a'', b'', c''; x)$, a contradiction.

Case 9. $-2x^{c+8} = -2x^{2a''+8}$ (the hypotheses are $c+6 = 2a+4, b > a+1$ and $2a''+4 = 2c''+2 < b''+3 = 2c''+4$). Then $2a+4 = c+6 = 2a''+6 = 2c''+4$ implies that $a = c''$, which is a contradiction to the fact that c'' is a positive odd number and a is a positive even number.

Therefore, except for the A-cospectral graphs $H(n; 2b, b, b)$ and $\Theta(b-2, 2b-3, b-1)$ with $b \geq 4$ being a positive even number, there is no graph $G_1 = H(n; c+6, a, b)$ being A-cospectral with $G_3 = \Theta(a'', b'', c'')$. This completes the proof of Lemma 5.11. □

Lemmas 5.4 and 5.11 imply the following result, since $H(4b; 2b, b, b)$ is the subdivision graph of $H(2b; b, \frac{b}{2}, \frac{b}{2})$ and $G_3 = \Theta(b-2, 2b-3, b-1)$ is the subdivision graph of G'_4 with $q' = b, d(u, v) = \frac{b}{2} - 1$ and $d(x, y) = \frac{b}{2} - 1$ (shown in Fig. 10).

Lemma 5.12. *Except for the graph $H(2b; b, \frac{b}{2}, \frac{b}{2})$ with $b \geq 4$ being a positive even number, there is no graph of the form $H(n; q, n_1, n_2)$ being Q-cospectral with graph G'_4. The graph $H(2b; b, \frac{b}{2}, \frac{b}{2})$ is Q-cospectral with graph G'_4 with $q' = b$, $d(u, v) = \frac{b}{2} - 1$ and $d(x, y) = \frac{b}{2} - 1$.*

In the following, we will prove that graphs $H(n; q, n_1, n_2)$, except for graphs $H(2a+4; a+3, \frac{a}{2}, \frac{a}{2}+1)$ with a being a positive even number and $H(2b; b, \frac{b}{2}, \frac{b}{2})$ with $b \geq 4$ being a positive even number, are determined by their Q-spectra. Before this, we give some useful lemmas.

Lemma 5.13. *Let $G = H(n; q, n_1, n_2)$. Then $\lambda_2(\mathcal{S}(G)) < 2$.*

Proof. Let u be the vertex of degree 4 in G. By the interlacing theorem for the A-spectrum, we obtain that

$$\lambda_2(\mathcal{S}(G)) \leq \lambda_1(\mathcal{S}(G) - u) = \lambda_1\left(P_{2q-1} \bigcup P_{2n_1} \bigcup P_{2n_2}\right) < 2,$$

since the largest eigenvalue for the A-spectrum of a path is less than 2. □

Lemma 5.14. *Let $G = H(n; q, n_1, n_2)$. If a graph H with $\deg(H) = (4, 2^{n-3}, 1^2)$ is Q-cospectral to G, then H does not contain cycles as its components.*

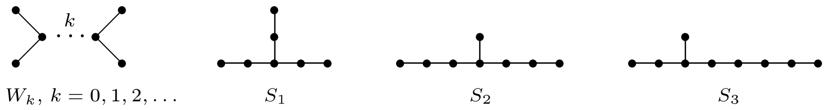

Fig. 7. Smith graphs W_k, S_1, S_2 and S_3.

Proof. $\mathcal{S}(H)$ is A-cospectral to $\mathcal{S}(G)$, since H is Q-cospectral to G. By Lemma 5.13, $\lambda_2(\mathcal{S}(H)) = \lambda_2(\mathcal{S}(G)) < 2$. By contradiction, we assume that $H = H' \bigcup C_k$. Then $\mathcal{S}(H) = \mathcal{S}(H') \bigcup C_{2k}$. Note that $\lambda_1(\mathcal{S}(H)) = \lambda_1(\mathcal{S}(G)) > 2$, since C_{2q} is a proper subgraph of $\mathcal{S}(G)$ and $\lambda_1(C_{2q}) = 2$. Then $\lambda_1(\mathcal{S}(H)) = \lambda_1(\mathcal{S}(H')) > 2$ and $\lambda_2(\mathcal{S}(H)) = \max\{\lambda_2(\mathcal{S}(H')), \lambda_1(C_{2q})\} \geq 2$, a contradiction. □

Recall that a connected graph which satisfies $\lambda_1 = 2$ is called a Smith graph (see [22]). These graphs are a cycle C_n ($n = 3, 4, \ldots$), and the graphs depicted in Fig. 7 (k denotes the length of the corresponding path in W_k, for $k = 0$ the graph reduces to $W_0 = K_{1,4}$).

Lemma 5.15. *Let $G = H(n; q, n_1, n_2)$. If a graph H is Q-cospectral to G, then H does not contain an induced subgraph isomorphic to the disjoint union of two cycles.*

Proof. $\mathcal{S}(H)$ is A-cospectral to $\mathcal{S}(G)$, since H is Q-cospectral to G. By Lemma 5.13, $\lambda_2(\mathcal{S}(H)) = \lambda_2(\mathcal{S}(G)) < 2$. It implies that $\mathcal{S}(H)$ has no induced subgraph isomorphic to the disjoint union of two cycles, since the largest eigenvalue for the A-spectrum of a cycle is 2. Therefore, graph H has no induced subgraph isomorphic to the disjoint union of two cycles. □

Similarly, we get the following lemma, since the subdivision graph of $K_{1,3}$ is S_1.

Lemma 5.16. *Let $G = H(n; q, n_1, n_2)$. If a graph H is Q-cospectral to G, then H does not contain an induced subgraph isomorphic to the disjoint union of a cycle and $K_{1,3}$.*

Theorem 5.17. *Let $G = H(n; q, n_1, n_2)$. Then graph G is determined by its Q-spectrum, except for graphs $H(2a+4; a+3, \frac{a}{2}, \frac{a}{2}+1)$ with a being a positive even number and $H(2b; b, \frac{b}{2}, \frac{b}{2})$ with $b \geq 4$ being an even number.*

Proof. Suppose that graphs G' and G are Q-cospectral. Then Lemma 2.9 implies that G' has n vertices, n edges and $\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} d_i'^2$, where d_i, d_i' are degrees of vertices v_i, v_i' in $H(n; q, n_1, n_2)$ and G', respectively. Suppose that G' has x_j' vertices of degree j, for $j = 0, 1, \ldots, \Delta$, where Δ is the maximum degree of G'. Then

$$\sum_{j=0}^{\Delta} x_j' = n, \tag{5.20}$$

$$\sum_{j=0}^{\Delta} jx_j' = 2n, \tag{5.21}$$

$$\sum_{j=0}^{\Delta} j^2x_j' = 16 + 4(n-3) + 2. \tag{5.22}$$

Therefore

$$\sum_{j=0}^{\Delta} (j^2 - 3j + 2)x_j' = 6, \tag{5.23}$$

i.e.,

$$2x_0' + 2x_3' + 6x_4' + 12x_5' + \sum_{j=6}^{\Delta} (j^2 - 3j + 2)x_j' = 6. \tag{5.24}$$

Eq. (5.24) implies that

Case 1. $x_0' = 3, x_3' = x_4' = \cdots = x_\Delta' = 0$. By (5.20) and (5.21), we have $x_1' = -6 < 0, x_2' = n + 3 > n$, a contradiction.

Case 2. $x_0' = 2, x_3' = 1, x_4' = \cdots = x_\Delta' = 0$. By (5.20) and (5.21), we have $x_1' = -3 < 0, x_2' = n$, a contradiction.

Case 3. $x_0' = 1, x_3' = 2, x_4' = \cdots = x_\Delta' = 0$. By (5.20) and (5.21), we have $x_1' = 0, x_2' = n - 3$. By (iv) of Lemma 2.9, we have

$$6n_G(C_3) + \sum_{i=1}^{n} d_i(G)^3 = 6n_{G'}(C_3) + \sum_{i=1}^{n} d_i(G')^3.$$

Then $n_{G'}(C_3) = n_G(C_3) + 2$.

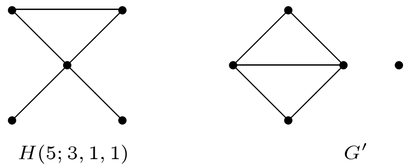

Fig. 8. Graphs $H(5; 3, 1, 1)$ and G'.

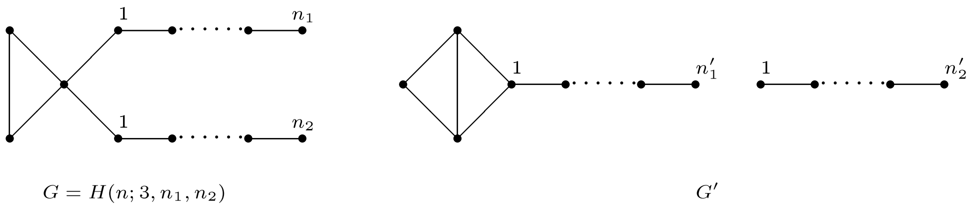

Fig. 9. Graphs $G = H(n; 3, n_1, n_2)$ and G'.

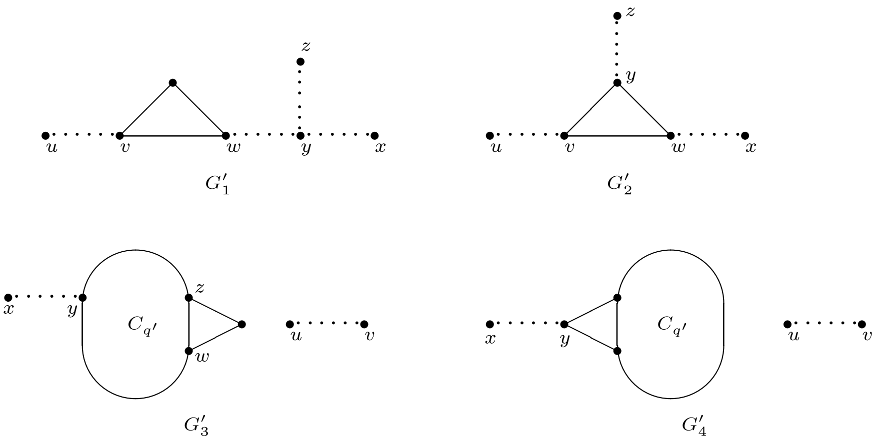

Fig. 10. Graphs G'_1, G'_2, G'_3 and G'_4.

Case 3.1. $n_G(C_3) = 1$. Then $n_{G'}(C_3) = 3$. Together with $\deg(G') = (3^2, 2^{n-3}, 0^1)$, there always exists an induced subgraph isomorphic to the disjoint union of two cycles, a contradiction to Lemma 5.15.

Case 3.2. $n_G(C_3) = 0$. Then $n_{G'}(C_3) = 2$. By Lemmas 5.15 and 5.16, the only possible case of G' is shown in Fig. 8. But it needs G to be $H(5; 3, 1, 1)$ (shown in Fig. 8), a contradiction to $n_G(C_3) = 0$.

Case 4. $x'_0 = 0, x'_3 = 3, x'_4 = \cdots = x'_\Delta = 0$. By (5.20) and (5.21), we have $x'_1 = 3, x'_2 = n - 6$. By (iv) of Lemma 2.9, we have

$$6n_G(C_3) + \sum_{i=1}^{n} d_i(G)^3 = 6n_{G'}(C_3) + \sum_{i=1}^{n} d_i(G')^3.$$

Then $n_{G'}(C_3) = n_G(C_3) + 1$. Together with $\deg(G') = (3^3, 2^{n-6}, 1^3)$, consider the following subcases.

Case 4.1. $n_G(C_3) = 1$. Then $n_{G'}(C_3) = 2$. By Lemmas 5.15 and 5.16, the only case of G' is shown in Fig. 9. Note that for the Q-spectrum the multiplicity of 0 gives the number of bipartite components [7]. Then there is an eigenvalue 0 in the Q-spectrum of G', but there is no eigenvalue 0 in the Q-spectrum of G. Clearly, G and G' are not Q-cospectral, a contradiction.

Case 4.2. $n_G(C_3) = 0$, i.e, $G = H(n; q, n_1, n_2)$ with $q \geq 4$. Then $n_{G'}(C_3) = 1$. By Lemmas 5.15 and 5.16, the possible cases of G' are shown in Fig. 10. Let $d(w, y)$ be the distance (the length of a shortest path) between two vertices w and y. Consider the following subcases.

Case 4.2.1. For graph G'_1, if $d(w, y) \geq 2$, then there always exists an induced subgraph isomorphic to the disjoint union of a cycle C_6 and Smith graph S_1 in $\mathcal{S}(G'_1)$, a contradiction to $\lambda_2(\mathcal{S}(G)) = \lambda_2(\mathcal{S}(G'_1)) < 2$.

If $d(w, y) = 1, d(y, z) \geq 3$ and $d(u, v) \geq 2$, or $d(w, y) = 1, d(y, x) \geq 3$ and $d(u, v) \geq 2$, then there always exists an induced subgraph isomorphic to the disjoint union of Smith graphs S_2 and S_3 in $\mathcal{S}(G'_1)$, a contradiction to $\lambda_2(\mathcal{S}(G)) = \lambda_2(\mathcal{S}(G'_1)) < 2$.

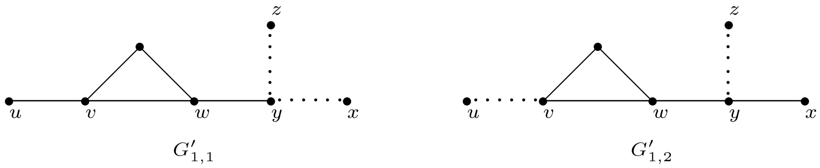

Fig. 11. Graphs $G'_{1,1}$ and $G'_{1,2}$.

If $d(w, y) = 1, d(y, x) \geq 2, d(y, z) \geq 2$, and $d(u, v) \geq 2$, then there always exists an induced subgraph isomorphic to the disjoint union of two Smith graphs S_2 in $\mathcal{S}(G'_1)$, a contradiction to $\lambda_2(\mathcal{S}(G)) = \lambda_2(\mathcal{S}(G'_1)) < 2$.

Then the two possible forms of G'_1 are shown in Fig. 11.

Case 4.2.1.1. For graph $G'_{1,1}$, Lemma 5.3 implies that line graphs $\mathcal{L}(G)$ and $\mathcal{L}(G'_{1,1})$ are A-cospectral. In the following, we use Lemma 2.4 to compute the number of closed walks of length 5 in $\mathcal{L}(G)$ and $\mathcal{L}(G'_{1,1})$, respectively.

For graph $\mathcal{L}(G)$,

$$N_{\mathcal{L}(G)}(5) = \begin{cases} 380, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 2 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 24, \\ \quad \text{if } q = 4,\ n_1,\ n_2 \geq 2, \\ 350, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 2 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 21, \\ \quad \text{if } q = 4,\ n_1 = 1, \text{ or } n_2 = 1, \\ 320, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 2 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 18, \\ \quad \text{if } q = 4,\ n_1 = n_2 = 1, \\ 370, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 1 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 24, \\ \quad \text{if } q = 5,\ n_1,\ n_2 \geq 2, \\ 340, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 1 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 21, \\ \quad \text{if } q = 5,\ n_1 = 1, \text{ or } n_2 = 1, \\ 310, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 1 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 18, \\ \quad \text{if } q = 5,\ n_1 = n_2 = 1, \\ 360, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 0 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 24, \\ \quad \text{if } q \geq 6,\ n_1,\ n_2 \geq 2, \\ 330, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 0 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 21, \\ \quad \text{if } q \geq 6,\ n_1 = 1, \text{ or } n_2 = 1, \\ 300, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 0 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 18, \\ \quad \text{if } q \geq 6,\ n_1 = n_2 = 1. \end{cases}$$

For graph $\mathcal{L}(G'_{1,1})$,

$$N_{\mathcal{L}(G'_{1,1})}(5) = \begin{cases} 290, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 1 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 16, \\ \quad \text{if } d(y,z),\ d(y,x) \geq 2, \\ 280, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 1 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 15, \\ \quad \text{if } d(y,z) = 1, \text{ or } d(y,x) = 1, \\ 270, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 1 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 14, \\ \quad \text{if } d(y,z) = d(y,x) = 1. \end{cases}$$

Clearly, $N_{\mathcal{L}(G)}(5) \neq N_{\mathcal{L}(G'_{1,1})}(5)$, a contradiction to (v) of Lemma 2.2.

Case 4.2.1.2. For graph $G'_{1,2}$, use the same method as in Case 4.2.1.1, we have

$$N_{\mathcal{L}(G'_{1,2})}(5) = \begin{cases} 290, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 1 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 16, \\ \quad \text{if } d(u,v),\ d(y,z) \geq 2, \\ 280, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 1 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 15, \\ \quad \text{if } d(u,v) = 1, \text{ or } d(y,z) = 1, \\ 270, \text{ since } n_{\mathcal{L}(G)}(K_3) = 4, & n_{\mathcal{L}(G)}(C_5) = 1 \text{ and } n_{\mathcal{L}(G)}(L(3,1)) = 14, \\ \quad \text{if } d(u,v) = d(y,z) = 1. \end{cases}$$

Clearly, $N_{\mathcal{L}(G)}(5) \neq N_{\mathcal{L}(G'_{1,2})}(5)$, a contradiction to (v) of Lemma 2.2.

Case 4.2.2. For graph G'_2, Lemma 5.9 implies that graph $H(2a + 4; a + 3, \frac{a}{2}, \frac{a}{2} + 1)$ is Q-cospectral with graph G'_2 with $d(z, y) = \frac{a}{2}, d(u, v) = \frac{a}{2}$ and $d(w, x) = a + 1$, where a is a positive even number.

Case 4.2.3. For graph G'_3, if $d(y, z) \geq 2, d(w, y) \geq 2$ and $q' \geq 5$, then there always exists an induced subgraph isomorphic to the disjoint union of a cycle C_6 and a Smith graph S_1 in $\mathcal{S}(G'_3)$, a contradiction to $\lambda_2(\mathcal{S}(G)) = \lambda_2(\mathcal{S}(G'_3)) < 2$.

If $d(y, z) = 1, d(x, y) \geq 1$ and $q' \geq 7$, then there always exists an induced subgraph isomorphic to the disjoint union of two Smith graphs S_2 and S_3 in $\mathcal{S}(G'_3)$, a contradiction to $\lambda_2(\mathcal{S}(G)) = \lambda_2(\mathcal{S}(G'_3)) < 2$.

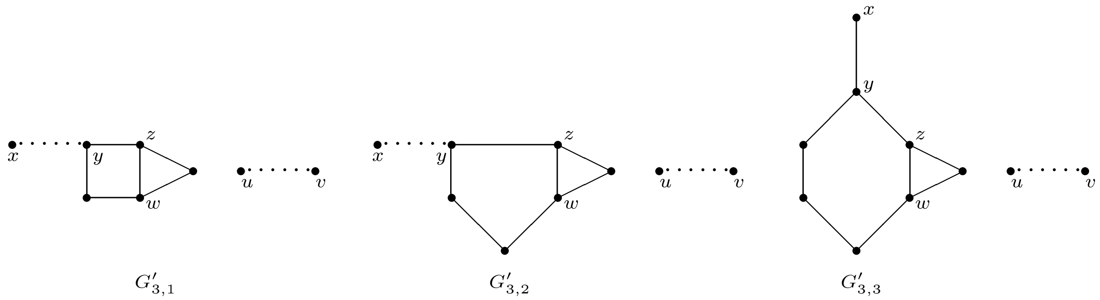

Fig. 12. Graphs $G'_{3,1}$, $G'_{3,2}$ and $G'_{3,3}$.

If $d(y,z)=1$, $d(x,y)\geq 2$ and $q'\geq 6$, then there always exists an induced subgraph isomorphic to the disjoint union of two Smith graphs S_2 in $\mathcal{S}(G'_3)$, a contradiction to $\lambda_2(\mathcal{S}(G))=\lambda_2(\mathcal{S}(G'_3))<2$.

Then the three possible forms of G'_3 are shown in Fig. 12.

Case 4.2.3.1. For graph $G'_{3,1}$, Lemma 5.3 implies that line graphs $\mathcal{L}(G)$ and $\mathcal{L}(G'_{3,1})$ are A-cospectral. By Lemma 2.3, we have $\sum_i \lambda_i(\mathcal{L}(G'_{3,1}))^4=\sum_i \lambda_i(\mathcal{L}(G))^4$. For graph $\mathcal{L}(G)$,

$$\sum_i \lambda_i(\mathcal{L}(G))^4=\begin{cases}6n+110, & \text{if } q=4,\ n_1,\ n_2\geq 2,\\ 6n+102, & \text{if } q=4,\ n_1=1, \text{ or } n_2=1,\\ 6n+94, & \text{if } q=4,\ n_1=n_2=1,\\ 6n+102, & \text{if } q\geq 5,\ n_1,\ n_2\geq 2,\\ 6n+94, & \text{if } q\geq 5,\ n_1=1, \text{ or } n_2=1,\\ 6n+86, & \text{if } q\geq 5,\ n_1=n_2=1.\end{cases}$$

By computing $\sum_i \lambda_i(\mathcal{L}(G'_{3,1}))^4$, we get

$$\sum_i \lambda_i(\mathcal{L}(G'_{3,1}))^4=\begin{cases}6n+98, & \text{if } d(u,v)\geq 2,\ d(x,y)\geq 2,\\ 6n+102, & \text{if } d(u,v)=1,\ d(x,y)\geq 2,\\ 6n+94, & \text{if } d(u,v)\geq 2,\ d(x,y)=1,\\ 6n+98, & \text{if } d(u,v)=1,\ d(x,y)=1.\end{cases}$$

Then consider the following equality cases:

Case 4.2.3.1.1. $\underline{6n+102}$: $H(n;4,n_1,n_2)$ with $n_1=1$, or $n_2=1$ and $G'_{3,1}$ with $d(u,v)=1$ and $d(x,y)\geq 2$. By Lemma 2.4, $N_{\mathcal{L}(G'_{3,1})}(5)=330\neq N_{\mathcal{L}(H(n;4,n_1,n_2))}(5)=350$, a contradiction to (v) of Lemma 2.2.

Case 4.2.3.1.2. $\underline{6n+102}$: $H(n;q,n_1,n_2)$ with $q\geq 5$ and $n_1,\ n_2\geq 2$ and $G'_{3,1}$ with $d(u,v)=1$ and $d(x,y)\geq 2$. Now, $N_{\mathcal{L}(G'_{3,1})}(5)=330\neq N_{\mathcal{L}(H(n;5,n_1,n_2))}(5)=370$, and

$$N_{\mathcal{L}(G'_{3,1})}(5)=330\neq N_{\mathcal{L}(H(n;q,n_1,n_2))}(5)=360,\quad \text{with } q\geq 6,$$

contradictions to (v) of Lemma 2.2.

Case 4.2.3.1.3. $\underline{6n+94}$: $H(n;4,n_1,n_2)$ with $n_1=n_2=1$ and $G'_{3,1}$ with $d(u,v)\geq 2$ and $d(x,y)=1$. Now, the number of vertices of $H(n;4,n_1,n_2)$ with $n_1=n_2=1$ is 6, but there are at least 9 vertices in $G'_{3,1}$ with $d(u,v)\geq 2$ and $d(x,y)=1$, a contradiction.

Case 4.2.3.1.4. $\underline{6n+94}$: $H(n;q,n_1,n_2)$ with $q\geq 5$ and $n_1=1$, or $n_2=1$ and $G'_{3,1}$ with $d(u,v)\geq 2$ and $d(x,y)=1$. By Lemma 2.4, $N_{\mathcal{L}(G'_{3,1})}(5)=320\neq N_{\mathcal{L}(H(n;5,n_1,n_2))}(5)=340$, and

$$N_{\mathcal{L}(G'_{3,1})}(5)=320\neq N_{\mathcal{L}(H(n;q,n_1,n_2))}(5)=330,\quad \text{with } q\geq 6,$$

contradictions to (v) of Lemma 2.2.

Case 4.2.3.2. For graph $G'_{3,2}$, Lemma 5.3 implies that line graphs $\mathcal{L}(G)$ and $\mathcal{L}(G'_{3,2})$ are A-cospectral. Similarly to Case 4.2.3.1, we compute $\sum_i \lambda_i(\mathcal{L}(G'_{3,2}))^4$. Then

$$\sum_i \lambda_i(\mathcal{L}(G'_{3,2}))^4=\begin{cases}6n+90, & \text{if } d(u,v)\geq 2,\ d(x,y)\geq 2,\\ 6n+94, & \text{if } d(u,v)=1,\ d(x,y)\geq 2,\\ 6n+86, & \text{if } d(u,v)\geq 2,\ d(x,y)=1,\\ 6n+90, & \text{if } d(u,v)=1,\ d(x,y)=1.\end{cases}$$

Then consider the following equality cases:

Case 4.2.3.2.1. $\underline{6n+94}$: $H(n;4,n_1,n_2)$ with $n_1=n_2=1$ and $G'_{3,2}$ with $d(u,v)=1$ and $d(x,y)\geq 2$. Now, the number of vertices of $H(n;4,n_1,n_2)$ with $n_1=n_2=1$ is 6, but there are at least 10 vertices in $G'_{3,2}$ with $d(u,v)=1$ and $d(x,y)\geq 2$, a contradiction.

Case 4.2.3.2.2. $\underline{6n+94}$: $H(n;q,n_1,n_2)$ with $q\geq 5$ and $n_1=1$ or $n_2=1$ and $G'_{3,2}$ with $d(u,v)=1$, $d(x,y)\geq 2$. By Lemma 2.4, $N_{\mathcal{L}(G'_{3,2})}(5)=310\neq N_{\mathcal{L}(H(n;5,n_1,n_2))}(5)=340$, and

$$N_{\mathcal{L}(G'_{3,2})}(5)=310\neq N_{\mathcal{L}(H(n;q,n_1,n_2))}(5)=330,\quad \text{with } q\geq 6,$$

contradictions to (v) of Lemma 2.2.

Case 4.2.3.2.3. $\underline{6n+86}$: $H(n;q,n_1,n_2)$ with $q\geq 5$ and $n_1=n_2=1$ and $G'_{3,2}$ with $d(u,v)\geq 2$, $d(x,y)=1$. Lemma 2.4 implies that $N_{\mathcal{L}(G'_{3,2})}(5)=300\neq N_{\mathcal{L}(H(n;5,n_1,n_2))}(5)=310$, a contradiction to (v) of Lemma 2.2.

For $H(n;q,n_1,n_2)$ with $q\geq 6$ and $n_1=n_2=1$, we use Lemma 2.4 to compute the number of closed walks of length 7 in $\mathcal{L}(H(n;q,n_1,n_2))$, then

$$N_{\mathcal{L}(G)}(7)=\begin{cases}3248, & \text{if } q=6,\\ 3234, & \text{if } q=7,\\ 3220, & \text{if } q\geq 8.\end{cases}$$

And the number of closed walks of length 7 in $\mathcal{L}(G'_{3,2})$ is $N_{\mathcal{L}(G'_{3,2})}(7)=3234$. Clearly, the unique equality case is graphs $H(n;7,n_1,n_2)$ with $n_1=n_2=1$ and $G'_{3,2}$ with $d(u,v)\geq 2$ and $d(x,y)=1$. Now, the number of vertices of $H(n;7,n_1,n_2)$ with $n_1=n_2=1$ is 9, but there are at least 10 vertices of $G'_{3,2}$ with $d(u,v)\geq 2$ and $d(x,y)=1$, a contradiction.

Case 4.2.3.3. For graph $G'_{3,3}$, Lemma 5.3 implies that line graphs $\mathcal{L}(G)$ and $\mathcal{L}(G'_{3,3})$ are A-cospectral. Then by Lemma 2.4, we have $N_{\mathcal{L}(G'_{3,3})}(5)=290$, since $n_{\mathcal{L}(G'_{3,3})}(K_3)=4$, $n_{\mathcal{L}(G'_{3,3})}(C_5)=1$ and $n_{\mathcal{L}(G'_{3,3})}(L(3,1))=16$. Clearly, $N_{\mathcal{L}(G'_{3,3})}(5)\neq N_{\mathcal{L}(G)}(5)$, a contradiction to (v) of Lemma 2.2.

Case 4.2.4. For graph G'_4, Lemma 5.12 implies that graph $H(2b;b,\frac{b}{2},\frac{b}{2})$ is Q-cospectral with graph G'_4, with $q'=b$, $d(u,v)=\frac{b}{2}-1$ and $d(x,y)=\frac{b}{2}-1$, where $b\geq 4$ is an even number.

Case 5. $x'_4=1$, $x'_0=x'_3=\cdots=x'_\Delta=0$. By (5.20) and (5.21), we have $x'_1=2$, $x'_2=n-3$. Then $\deg(G')=(4,2^{n-3},1^2)$. By Lemma 5.14, G' contains no cycles as its components. Then G' is the form of either Γ_2 or G.

Case 5.1. G' is the form of Γ_2, consider their subdivision graphs $\mathcal{S}(H(n;q,n_1,n_2))$ and $\mathcal{S}(\Gamma_2)$. Lemma 5.4 implies that $\mathcal{S}(H(n;q,n_1,n_2))$ and $\mathcal{S}(\Gamma_2)$ are A-cospectral. But by Lemma 5.2, $\mathcal{S}(H(n;q,n_1,n_2))$ and $\mathcal{S}(\Gamma_2)$ cannot be A-cospectral, a contradiction.

Case 5.2. G' is the form of G. Suppose that $G'=H(n;q',n'_1,n'_2)$. Lemma 5.4 implies that $\mathcal{S}(H(n;q,n_1,n_2))$ and $\mathcal{S}(H(n;q',n'_1,n'_2))$ are A-cospectral. By Lemma 5.5, $\mathcal{S}(H(n;q,n_1,n_2))$ and $\mathcal{S}(H(n;q',n'_1,n'_2))$ are isomorphic.

Therefore, $H(n;q,n_1,n_2)$ and G' are isomorphic, except for graphs $H(2a+4;a+3,\frac{a}{2},\frac{a}{2}+1)$ with a being a positive even number and $H(2b;b,\frac{b}{2},\frac{b}{2})$ with $b\geq 4$ being an even number. This completes the proof of Theorem 5.17. □

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References

[1] N.L. Biggs, Algebraic Graph Theory, second edition, Cambridge University Press, 1993.
[2] R. Boulet, B. Jouve, The lollipop graph is determined by its spectrum, Electron. J. Combin. 15 (2008) R74.
[3] R.A. Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge U. P., New York, 1991.
[4] D.M. Cvetković, New theorems for signless Laplacians eigenvalues, Bull. Acad. Serbe Sci. Arts, Cl. Sci. Math. Natur., Sci. Math. 137 (33) (2008) 131–146.
[5] D.M. Cvetković, M. Doob, H. Sachs, Spectral of Graphs Theory and Applications, Academic press, New York, San Francisco, London, 1980.
[6] D.M. Cvetković, P. Rowlinson, Spectra of unicyclic graph, Graphs Combin. 3 (1987) 7–23.
[7] D.M. Cvetković, P. Rowlinson, S.K. Simić, Signless Laplacians of finite graphs, Linear Algebra Appl. 423 (2007) 155–171.
[8] D.M. Cvetković, S.K. Simić, Towards a spectral theory of graphs based on signless Laplacian, I, Publ. Inst. Math. (Beograd) (N.S.) 85 (99) (2009) 1–15.
[9] F.R. Gantmacher, The Theory of Matrices, vol. II, Chelsea, New York, 1959.
[10] C.D. Godsil, B.D. McKay, Constructing cospectral graphs, Aequationes Math. 25 (1982) 257–268.
[11] W.H. Haemers, X.-G. Liu, Y.-P. Zhang, Spectral characterizations of lollipop graphs, Linear Algebra Appl. 428 (2008) 2415–2423.
[12] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.
[13] A.K. Kelmans, The number of trees in a graph I, II, Autom. Remote Control 26 (1965) 2118–2129. 27 (1966) 233–241, translated from Avtomatika i Telemekhanika 26 (1965) 2194–2204 and 27 (1966) 56–65 [in Russian].
[14] A.K. Kelmans, V.M. Chelnokov, A certain polynomial of a graph and graphs with an extremal number of trees, J. Combin. Theory, Ser. B 16 (1974) 197–214.
[15] J.-S. Li, Y.-L. Pan, A note on the second largest eigenvalue of the Laplacian matrix of a graph, Linear Multilinear Algebra 48 (20) (2000) 117–121.
[16] X.-G. Liu, Y.-P. Zhang, X.-Q. Gui, The multi-fan graphs are determined by their Laplacian spectra, Discrete Math. 308 (2008) 4267–4271.

[17] J.-S. Li, X.-D. Zhang, On the Laplacian eigenvalues of a graph, Linear Algebra Appl. 285 (1998) 305–307.
[18] C.S. Oliveira, N.M.M. de Abreu, S. Jurkiewilz, The characteristic polynomial of the Laplacian of graphs in (a, b)-linear cases, Linear Algebra Appl. 365 (2002) 113–121.
[19] G.R. Omidi, On a signless Laplacian spectral characterizaiton of T-shape trees, Linear Algebra Appl. 431 (2009) 1607–1615.
[20] G.R. Omidi, K. Tajbakhsh, Starlike trees are determined by their Laplacian spectrum, Linear Algebra Appl. 422 (2007) 654–658.
[21] S.K. Simić, Z. Stanić, Q-integral graphs with edge-degrees at most five, Discrete Math. 308 (2008) 4625–4634.
[22] J.H. Smith, Some properties of spectrum of graphs, in: R. Guy, H. Hanani, N. Sauer, J. Schönheim (Eds.), Combinatorial Structures and their Applications, Gordon and Breach, Science Publ., Inc., New York, London, Paris, 1970, pp. 403–406.
[23] E.R. van Dam, W.H. Haemers, Which graphs are determined by their spectrum? Linear Algebra Appl. 373 (2003) 241–272.
[24] E.R. van Dam, W.H. Haemers, Developments on spectral characterizations of graphs, Discrete Math. 309 (2009) 576–586.
[25] J.-F. Wang, Q.-X. Huang, F. Belardo, E.M. Li Marzi, On the spectral characterizations of ∞-graphs, Discrete Math. 310 (2010) 1845–1855.
[26] W. Wang, C.-X. Xu, Note: the T-shape tree is determined by its Laplacian spectrum, Linear Algebra Appl. 419 (2006) 78–81.