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The asymptotic number of spanning trees in circulant graphs[☆]

Mordecai J. Golin[a], Xuerong Yong[b], Yuanping Zhang[c]

[a] Dept. of CSE, Hong Kong UST, Kowloon, Hong Kong
[b] Dept. of Mathematical Sciences, Univ. of Puerto Rico, Mayaguez, United States
[c] School of Computer and Communication, Lanzhou University of Technology, Gansu, PR China

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ABSTRACT

Let $T(G)$ be the number of spanning trees in graph G. In this note, we explore the asymptotics of $T(G)$ when G is a circulant graph with given jumps.

The circulant graph $C_n^{s_1,s_2,\ldots,s_k}$ is the $2k$-regular graph with n vertices labeled $0, 1, 2, \ldots, n-1$, where node i has the $2k$ neighbors $i \pm s_1, i \pm s_2, \ldots, i \pm s_k$ where all the operations are $(\mathrm{mod}\ n)$. We give a closed formula for the asymptotic limit $\lim_{n\to\infty} T(C_n^{s_1,s_2,\ldots,s_k})^{\frac{1}{n}}$ as a function of $s_1, s_2, \ldots, s_k$. We then extend this by permitting some of the jumps to be linear functions of n, i.e., letting s_i, d_i and e_i be arbitrary integers, and examining

$$\lim_{n\to\infty} T\left(C_n^{s_1,s_2,\ldots,s_k,\lfloor\frac{n}{d_1}\rfloor+e_1,\lfloor\frac{n}{d_2}\rfloor+e_2,\ldots,\lfloor\frac{n}{d_l}\rfloor+e_l}\right)^{\frac{1}{n}}.$$

While this limit does not usually exist, we show that there is some p such that for $0 \le q < p$, there exists c_q such that limit (1) restricted to only n congruent to q modulo p does exist and is equal to c_q. We also give a closed formula for c_q.

One further consequence of our derivation is that if s_i go to infinity (in any arbitrary order), then

$$\lim_{s_1,s_2,\ldots,s_k\to\infty}\ \lim_{n\to\infty} T(C_n^{s_1,s_2,\ldots,s_k})^{\frac{1}{n}} = 4\exp\left[\int_0^1\int_0^1\cdots\int_0^1 \ln\left(\sum_{i=1}^{k}\sin^2\pi x_i\right)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_k\right].$$

Interestingly, this value is the same as the asymptotic number of spanning trees in the k-dimensional square lattice recently obtained by Garcia, Noy and Tejel.

1. Introduction

Throughout this paper, we permit graphs (digraphs) to contain multiple edges (arcs) and self-loops unless otherwise specified. Let G and D denote a graph and a digraph, respectively. *A spanning tree* in G is a tree having the same vertex set as G. *An oriented spanning tree* in D is a rooted tree with the same vertex set as D, i.e., there is a specified root node and paths from it to every vertex of D. The study of the number of spanning trees in a graph has a long history. Evaluating this number is

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E-mail addresses: golin@cs.ust.hk (M.J. Golin), xryong@math.uprm.edu (X. Yong), ypzhang@lut.cn (Y. Zhang).

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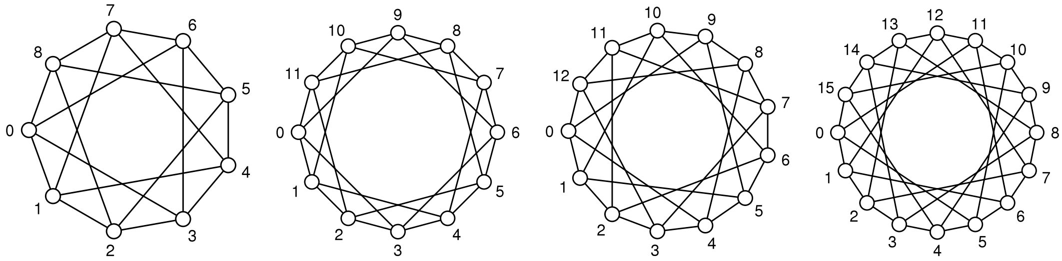

(a) $C_9^{1,3}$. (b) $C_{12}^{1,3}$. (c) $C_{13}^{1,4}$. (d) $C_{16}^{1,5}$.

Fig. 1. 4 circulant graphs. (b) and (d) are $C_{4n+8}^{1,2n+1}$ for $n = 1, 2$. (a) and (c) are $C_{4n+1}^{1,n+1}$ for $n = 2, 3$.

not only interesting from a combinatorial perspective but also arises in practical applications, e.g., analyzing the reliability of a network in the presence of line faults, designing electrical circuits etc. [8]. Given the adjacency matrix of G (or D), *Kirchoff's matrix tree theorem* [12] gives a closed formula for calculating the number of spanning trees. The real problem, then, is to calculate the number of spanning trees of graphs in particular parameterized classes, as a function of the parameters. A well studied class, which we will be further analyzing in this paper, is the *circulant graphs*.

We start by formally defining the graphs and the values to be counted. Let $s_1, s_2, \ldots, s_k$ be positive integers. The circulant graph with n vertices and jumps $s_1, s_2, \ldots, s_k$ is defined by

$$C_n^{s_1,s_2,\ldots,s_k} = (V, E)$$

where

$$V = \{0, 1, 2, \ldots, n-1\}, \quad \text{and} \quad E = \bigcup_{i=0}^{n-1} \{(i, i \pm s_1), (i, i \pm s_2), \ldots, (i, i \pm s_k)\}$$

where all of the additions are modulo n. That is, each node is connected to the nodes that are jumps $\pm s_j$ away from it, for $j = 1, 2, \ldots, k$.[1] Similarly the directed circulant graph, $\vec{C}_n^{s_1,s_2,\ldots,s_k}$, has the same vertex set, but

$$E = \bigcup_{i=0}^{n-1} \{(i, i + s_1), (i, i + s_2), \ldots, (i, i + s_k)\}$$

i.e, there is an edge directed from each i to the nodes s_j ahead of it, for $j = 1, 2, \ldots, k$. Examples of four undirected circulant graphs are given in Fig. 1.

We will use $T(X)$ to denote the number of spanning trees in a directed or undirected graph X. It was shown in [17] that, for directed circulant graphs,

$$\lim_{n\to\infty} \frac{T(\vec{C}_{n+1}^{s_1,s_2,\ldots,s_k})}{T(\vec{C}_n^{s_1,s_2,\ldots,s_k})} = k,$$

where k is the degree of each vertex of $\vec{C}_{n+1}^{s_1,s_2,\ldots,s_k}$. One might hope that similar asymptotic behavior, i.e., a limit dependent only upon k but independent of the actual values of the s_i, would also be true for undirected circulant graphs. Unfortunately, as seen in the asymptotic (numerical) results presented in Table 1 of [18], this is not the case; the asymptotic limits do seem somehow dependent on the s_i.

We therefore, in that paper, posed "*the analysis of the asymptotics as a function of the s_i*" as an open question. This paper addresses that question.

The problem of calculating the asymptotic maximum number of spanning trees in a circulant graph with k jumps was treated in [13], but their technique does not seem to permit analyzing the number of spanning trees for any given fixed jumps. Asymptotic limits for grids and tori (which turn out to be equal) were obtained in [6,9]. More recently, while examining the structure of non-constant jump circulant graphs, it was conjectured in [10] that the asymptotics of the number of spanning trees of the $m \times n$ tori and grids and the circulant graphs $C_{mn}^{1,n}$ would be the same.

The main result of this paper is the derivation in Section 2 of closed formulas for the first order asymptotics of the number of spanning trees in undirected circulant graphs, both for fixed jump circulants and linear jump ones (in which the jump

[1] To avoid confusion, we emphasize that, since we are allowing multiple edges in our graphs, $C_n^{s_1,s_2,\ldots,s_k}$ is *always* $2k$-regular and $\vec{C}_n^{s_1,s_2,\ldots,s_k}$ is *always* k-regular. For example, in our notation, $C_{2n}^{1,n}$ is the 4-regular graph with $2n$ vertices such that each vertex i is connected by *one* edge to each of $(i-1) \bmod 2n$ and $(i+1) \bmod 2n$ and by *two* edges to $(i+n) \bmod 2n$. Our techniques would, with slight technical modifications, also permit analyzing graphs in which multiple edges are *not* allowed, e.g., the *Mobius ladder* M_{2n}. This is the 3-regular graph with $2n$ vertices such that each vertex i is connected by one edge to each of $(i-1) \bmod 2n$, $(i+1) \bmod 2n$ and $(i+n) \bmod 2n$. The reason that we do not explicitly analyze such graphs is that such an analysis would require rewriting all of our theorems a second time to deal with these special instances without introducing any new interesting techniques.

sizes can depend linearly upon n).[2] We note that the corresponding problem for directed circulants was recently addressed by [7].

A secondary result that follows from our primary ones is that, as described in Section 3, the limiting asymptotics of the number of spanning trees of circulant graphs $C_n^{s_1,s_2,\ldots,s_k}$ as $s_1, s_2, \ldots, s_k, n$ tend to infinity will be exactly the same as the limiting asymptotics of the number of spanning trees in the k-dimensional tori (and k-dimensional grids) when the number of vertices in the tori (and grids) tend to infinity.

Returning to our main result, it is not reasonable to assume *a priori* that

$$\lim_{n\to\infty} T\left(C_n^{s_1,s_2,\ldots,s_k,\lfloor \frac{n}{d_1}\rfloor+e_1,\lfloor \frac{n}{d_2}\rfloor+e_2,\ldots,\lfloor \frac{n}{d_l}\rfloor+e_l}\right)^{\frac{1}{n}} \tag{1}$$

exists. Consider, for example, the simple case of $C_n^{1,\lfloor \frac{n}{3}\rfloor}$, and partition the graphs by the congruence classes of n modulo 3.

$\mathbf{q = 0:}$ If $n = 3k$ then $C_n^{1,\lfloor \frac{n}{3}\rfloor}$ is the union of $k+1$ disjoint cycles; one of size $3k = n$ and $k = n/3$ of size 3.
$\mathbf{q = 1:}$ If $n = 3k+1$ then $C_n^{1,\lfloor \frac{n}{3}\rfloor}$ is the union of exactly two disjoint cycles, each of size n.
$\mathbf{q = 2:}$ If $n = 3k+2$,
- if k is odd, then $C_n^{1,\lfloor \frac{n}{3}\rfloor}$ is the union of exactly two disjoint cycles, each of size n,
- if k is even it is the union of 3 cycles; one of size $3k = n$ and 2 of size $n/2$.

It is unreasonable to expect that all three types of graphs have the same limiting behavior. In fact, they do not. Our results will imply though, that, for $q = 0, 1, 2$,

$$\lim_{\substack{n\to\infty \\ n \bmod 3=q}} T\left(C_n^{1,\lfloor \frac{n}{3}\rfloor}\right) = c_q$$

where the c_q are three different constants.

More specifically, in the next section (Theorem 4), we will show that, if $p = \mathrm{lcm}(d_1, d_2, \ldots, d_l)$, where lcm denotes the *least common multiple*, then for $0 \le q < p$,

$$\lim_{\substack{m\to\infty \\ m \bmod p=q}} T\left(C_m^{s_1,s_2,\ldots,s_k,\lfloor \frac{m}{d_1}\rfloor+e_1,\lfloor \frac{m}{d_2}\rfloor+e_2,\ldots,\lfloor \frac{m}{d_l}\rfloor+e_l}\right)^{\frac{1}{m}} = c_q,$$

and we will give a closed form for c_q in terms of the s_i, d_i, and e_i.

Most studies of the number of spanning trees in circulants start with the following facts. It is known [8] that the formula for the number of spanning trees in a d-regular graph G can be expressed as

$$T(G) = \frac{1}{n}\prod_{j=1}^{n-1}(d-\lambda_j), \tag{2}$$

where $\lambda_0 = d, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. Because the adjacency matrix of $C_n^{s_1,s_2,\ldots,s_k}$ is circulant, from [3] we have

$$\lambda_j = \varepsilon^{s_1 j} + \varepsilon^{s_2 j} + \cdots + \varepsilon^{s_k j} + \varepsilon^{-s_1 j} + \varepsilon^{-s_2 j} + \cdots + \varepsilon^{-s_k j}, \quad j = 0, 1, \ldots, n-1,$$

where $\varepsilon = e^{\frac{2\pi\sqrt{-1}}{n}}$. This fact directly implies the known result:

$$T(C_n^{s_1,s_2,\ldots,s_k}) = \frac{1}{n}\prod_{j=1}^{n-1}\left(2k - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{n}\right). \tag{3}$$

Starting from this, in [18] it was proved that

Lemma 1. *For any fixed integers* $1 \le s_1 \le s_2 \le \cdots \le s_k$,

$$T(C_n^{s_1,s_2,\ldots,s_k}) = na_n^2,$$

where the a_n *satisfy linear recurrence relations of order* 2^{s_k-1}. *Furthermore, the largest characteristic root (in modulus) of* a_n *is unique.*

(This lemma is actually a combination of Lemma 4 and Lemma 5 from [18] and the "Note" following Lemma 4. Technically, the results in [18] state that $1 \le s_1 < s_2 < \cdots < s_k$, but, strict inequality was never used in the proofs there, so the results hold for $1 \le s_1 \le s_2 \le \cdots \le s_k$.)

[2] We note that, recently, Lyons [15] has developed general techniques for deriving the asymptotics of the spanning trees of large graphs. His techniques can be used to derive the asymptotics of fixed jump circulants (our Lemma 2) but do not seem to be usable to derive the asymptotics when the jumps are not fixed constants.

Eq. (3) and the formula in Lemma 1 will be crucial for our later analysis.

Lemma 1 assumes fixed jumps. The number of spanning trees in non-fixed jump circulant graphs, $T\left(C_{pq+r}^{a_1n+b_1,a_2n+b_2,\dots,a_kn+b_k}\right)$, was also shown [11] to satisfy a linear fixed order recurrence relation but no theorem as strong as Lemma 1 is known for the non-fixed jump cases.

Lemma 1 actually provides an algorithmic way of determining the asymptotics of

$$\lim_{n\to\infty} T(C_n^{s_1,s_2,\dots,s_k})^{1/n} = \lim_{n\to\infty}\left(na_n^2\right)^{1/n} = \lim_{n\to\infty}\left(a_n^2\right)^{1/n}.$$

Recall that Kirchoff's Matrix Tree Theorem provides a closed formula for the number of spanning trees in any given graph. For fixed $s_1, s_2, \dots, s_k$ it can be used to evaluate $T(C_n^{s_1,s_2,\dots,s_k})$ for $n \le 2^{s_k}$ and calculate the corresponding a_n. Lemma 1 states that a_n satisfies a linear recurrence relation of order 2^{s_k-1}. The 2^{s_k} initial values then permit solving for the coefficients of the recurrence relation. Since the recurrence relation has a unique largest characteristic root, we can then derive the asymptotics of $\left(a_n^2\right)^{1/n}$ and thus $T(C_n^{s_1,s_2,\dots,s_k})^{1/n}$. In the non-fixed jump case, the results in [11] similarly permit deriving a recurrence relation and then the asymptotics.

As a simple example, consider the square cycle $C_n^{1,2}$. Using Lemma 1 it is not hard to derive that

$$T(C_n^{1,2}) = nF_n^2,$$

(this was originally conjectured by [2,5] and variously proven by [4,16,18]) where F_n is the *Fibonacci* sequence, i.e., $F_1 = F_2 = 1$, and for $n > 2$, $F_n = F_{n-1} + F_{n-2}$. This implies that

$$\lim_{n\to\infty} T(C_n^{1,2})^{1/n} = \lim_{n\to\infty}\frac{T(C_{n+1}^{1,2})}{T(C_n^{1,2})} = \lim_{n\to\infty}\frac{F_{n+1}^2}{F_n^2} = \frac{3+\sqrt{5}}{2}.$$

Note though, that we did not calculate $\frac{3+\sqrt{5}}{2}$, by plugging $s_1 = 1, s_2 = 2$ into a closed formula. Instead, we essentially used the fact that $T(C_n^{1,2})$ satisfied a recurrence relation to then derive the recurrence relation and then plugged in the asymptotics of the solution to the recurrence relation. In this paper, we show the existence of a simple formula in the s_i that yields the asymptotics.

2. Spanning trees in circulant graphs

The ultimate goal of this section is to analyze the following quantity,

$$\lim_{n\to\infty} T\left(C_n^{s_1,s_2,\dots,s_k,\lfloor\frac{n}{d_1}\rfloor+e_1,\lfloor\frac{n}{d_2}\rfloor+e_2,\dots,\lfloor\frac{n}{d_l}\rfloor+e_l}\right)^{\frac{1}{n}} \tag{4}$$

as a function of given integers s_i, d_i and e_i. We will do this in stages. Note that, from symmetry considerations, restricting s_i to be positive and $d_i > 1$ will not change the classes of circulants that we address, so we will implicitly make these assumptions.

Before starting, we need to note an important caveat, which is that all limits will be over non-zero values. More specifically, note that, if $gcd(n, s_1, \dots, s_k) > 1$, then $C_n^{s_1,s_2,\dots,s_k}$ is disconnected so it has no spanning trees. This makes it impossible for us to define a limit. For example, when n is even, $C_n^{2,4}$ has two components, so no spanning tree exists and $T(C_n^{2,4}) = 0$. On the other hand, when n is not even, $T(C_n^{2,4}) > 0$ and we can show the existence of $c > 0$ such that $\lim_{m\to\infty} T(C_{2m+1}^{2,4})^{1/(2m+1)} = c$. Thus, technically, $\lim_{n\to\infty} T(C_n^{2,4})$ does not exist. But, as mentioned, we will take all of our limits to be over non-zero values, so we will write $\lim_{n\to\infty} T(C_n^{2,4})^{1/n} = c$.

We first start by analyzing (4) when all of the jumps are constant, i.e., $l = 0$, and prove the following lemma. We should point out that, as mentioned, Lyons' [15] recent results also imply the following lemma. Our reason for giving an alternative proof is that we will apply the same techniques later in the paper to derive the formulas for the cases with some non-constant jumps.

Lemma 2. *For any fixed integers $1 \le s_1 \le s_2 \le \dots \le s_k$,*

$$\lim_{n\to\infty} T(C_n^{s_1,s_2,\dots,s_k})^{\frac{1}{n}} = \lim_{n\to\infty}\frac{T(C_{n+1}^{s_1,s_2,\dots,s_k})}{T(C_n^{s_1,s_2,\dots,s_k})} = 4\exp\left[\int_0^1 \ln\left(\sum_{i=1}^k \sin^2 \pi s_i x\right) \mathrm{d}x\right].$$

Proof. Write T_n for $T(C_n^{s_1,s_2,\dots,s_k})$. From Lemma 1, $T_n = na_n^2$ where $a_n = \alpha^n(1 + O(\epsilon^n))$ for some $\alpha > 1$ and $\epsilon < 1$. Set $\beta = \alpha^2$. Then

$$T_n = na_n^2 = n\alpha^{2n}\left(1 + O(\epsilon^n)\right)^2 = n\beta^n(1 + O(\epsilon^n)),$$

so

$$\lim_{n\to\infty}\frac{T_{n+1}}{T_n}=\beta=\lim_{n\to\infty}T_n^{1/n},$$

proving the first equality of the Lemma.

To prove the second equality, note that, from (3) and $1-\cos(2x)=2\sin^2 x$, we get

$$\begin{aligned}\lim_{n\to\infty}T_n{}^{\frac{1}{n}} &= \lim_{n\to\infty}\left[\frac{1}{n}\prod_{j=1}^{n-1}\left(2k-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{n}\right)\right]^{\frac{1}{n}}\\
&= \lim_{n\to\infty}\exp\left[\ln\left(\prod_{j=1}^{n-1}\left(2k-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{n}\right)\right)\times\frac{1}{n}\right]\\
&= 4\lim_{n\to\infty}\exp\left[\ln\left(\prod_{j=1}^{n-1}\left(\sum_{i=1}^{k}\sin^2\frac{\pi s_i j}{n}\right)\right)\times\frac{1}{n}\right]\\
&= 4\lim_{n\to\infty}\exp\left[\sum_{j=1}^{n-1}\ln\left(\sum_{i=1}^{k}\sin^2\frac{\pi s_i j}{n}\right)\times\frac{1}{n}\right].\end{aligned}$$

We can conclude by using the fact that, if $f(x)$ is a continuous non-negative real function defined on $(0,1]$ such that $\int_0^1\ln(f(x))dx$ exists, then

$$\lim_{n\to\infty}\left(\sum_{j=1}^{n-1}\ln\left(f\left(\frac{j}{n}\right)\right)\times\frac{1}{n}\right)=\int_0^1\ln(f(x))\mathrm{d}x,$$

to get

$$\lim_{n\to\infty}T_n{}^{\frac{1}{n}}=4\exp\left[\int_0^1\ln\left(\sum_{i=1}^{k}\sin^2\pi s_i x\right)\mathrm{d}x\right].\quad\square$$

We now derive the asymptotics of the simplest non-constant jump case:

Theorem 3. *Let $1\le s_1\le\cdots\le s_k$, p and $0\le a_1\le\cdots\le a_l<p$ be integers. Then*

$$\lim_{n\to\infty}T(C_{pn}^{s_1,\ldots,s_k,a_1n,\ldots,a_ln})^{\frac{1}{n}}=4^p\exp\left[\sum_{t=0}^{p-1}\int_0^1\ln\left(\sum_{i=1}^{k}\sin^2\pi s_i x+\sum_{i=1}^{l}\sin^2\frac{\pi a_i t}{p}\right)\mathrm{d}x\right].$$

(Note that if $p=1$, then $l=0$ and this theorem reduces to Lemma 2.)

Proof. By (3), we have

$$\begin{aligned}T(C_{pn}^{s_1,\ldots,s_k,a_1n,\ldots,a_ln}) &= \frac{1}{pn}\prod_{j=1}^{pn-1}\left[2(k+l)-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn}-2\sum_{i=1}^{l}\cos\frac{2\pi a_i nj}{pn}\right]\\
&= \frac{1}{pn}\prod_{\substack{j=1\\(j\bmod p)\neq 0}}^{pn-1}\left[2(k+l)-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn}-2\sum_{i=1}^{l}\cos\frac{2\pi a_i nj}{pn}\right]\\
&\quad\times\prod_{\substack{j=1\\(j\bmod p)=0}}^{pn-1}\left[2(k+l)-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn}-2\sum_{i=1}^{l}\cos\frac{2\pi a_i nj}{pn}\right]\\
&= \frac{1}{pn}\prod_{t=1}^{p-1}\left(\prod_{\substack{j=1\\(j\bmod p)=t}}^{pn-1}\left[2(k+l)-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn}-2\sum_{i=1}^{l}\cos\frac{2\pi a_i j}{p}\right]\right)\\
&\quad\times\prod_{j'=1}^{n-1}\left[2k-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j'}{n}\right].\end{aligned}$$

To evaluate the limit of the nth root of the second product recall that, in the proof of Lemma 2, we already saw that

$$\lim_{n\to\infty}\left[\prod_{j=1}^{n-1}\left(2k-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{n}\right)\right]^{\frac{1}{n}} = 4\exp\left[\int_0^1 \ln\left(\sum_{i=1}^{k}\sin^2\pi s_i x\right)\mathrm{d}x\right].$$

To evaluate the limit of the nth root of the first product note that if $(j \bmod p) = t \neq 0$ then $j = pj' + t$ for some j' and

$$\begin{aligned}
&\lim_{n\to\infty}\left(\prod_{\substack{j=1\\(j \bmod p)=t}}^{pn-1}\left[2(k+l)-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn}-2\sum_{i=1}^{l}\cos\frac{2\pi a_i nj}{pn}\right]\right)^{\frac{1}{n}}\\
&= \lim_{n\to\infty}\exp\left[\ln\left(\prod_{j'=1}^{n-1}\left[2\left(k+l-\sum_{i=1}^{l}\cos\frac{2t\pi a_i}{p}\right)-2\sum_{i=1}^{k}\cos 2\pi s_i\left(\frac{j'}{n}+\frac{t}{pn}\right)\right]\right)\times\frac{1}{n}\right]\\
&= \lim_{n\to\infty}\exp\left[\sum_{j'=1}^{n-1}\left(\ln\left[2\left(k+l-\sum_{i=1}^{l}\cos\frac{2t\pi a_i}{p}\right)-2\sum_{i=1}^{k}\cos 2\pi s_i\left(\frac{j'}{n}+\frac{t}{pn}\right)\right]\right)\times\frac{1}{n}\right].
\end{aligned}$$

Since

$$\cos 2\pi s_i\left(\frac{j'}{n}+\frac{t}{pn}\right)\to\cos 2\pi s_i\frac{j'}{n}$$

uniformly (in j') as $n\to\infty$, the exact same type of calculation as in the proof of Lemma 2 yields that the limit is

$$\begin{aligned}
&\exp\left[\int_0^1\ln\left(2\left[k+l-\sum_{i=1}^{l}\cos\frac{2t\pi a_i}{p}\right]-2\sum_{i=1}^{k}\cos 2\pi s_i x\right)\mathrm{d}x\right]\\
&\quad = 4\exp\left[\int_0^1\ln\left(\sum_{i=1}^{k}\sin^2\pi s_k x+\sum_{i=1}^{l}\sin^2\frac{\pi a_i t}{p}\right)\mathrm{d}x\right].
\end{aligned}$$

where the last equality again comes from $2\sin^2 x = 1-\cos 2x$.

Combining the above equations gives

$$\begin{aligned}
\lim_{n\to\infty} T(C_{pn}^{s_1,\ldots,s_k,a_1n,\ldots,a_ln})^{\frac{1}{n}} &= \lim_{n\to\infty}\left(\frac{1}{pn}\right)^{\frac{1}{n}} 4\exp\left[\int_0^1\ln\left(\sum_{i=1}^{k}\sin^2\pi s_i x\right)\mathrm{d}x\right]\\
&\quad\times\prod_{t=1}^{p-1}\left(4\exp\left[\int_0^1\ln\left(\sum_{i=1}^{k}\sin^2\pi s_i x+\sum_{i=1}^{l}\sin^2\frac{\pi a_i t}{p}\right)\mathrm{d}x\right]\right)\\
&= 4^p\exp\left[\sum_{t=0}^{p-1}\int_0^1\ln\left(\sum_{i=1}^{k}\sin^2\pi s_i x+\sum_{i=1}^{l}\sin^2\frac{\pi a_i t}{p}\right)\mathrm{d}x\right]. \quad\square
\end{aligned}$$

We can now extend this to the case where the number of vertices in the graph is no longer an exact multiple of p.

Corollary 1. *Let $1\le s_1\le\cdots\le s_k$ and $0\le a_1\le\cdots\le a_l<p$ be integers and let q be an integer such that $0<q<p$. Then*

$$\lim_{n\to\infty} T\left(C_{pn+q}^{s_1,\ldots,s_k,a_1n,\ldots,a_ln}\right)^{\frac{1}{n}} = 4^p\exp\left[\sum_{t=0}^{p-1}\int_0^1\ln\left(\sum_{i=1}^{k}\sin^2\pi s_k x+\sum_{i=1}^{l}\sin^2\pi a_i\left(\frac{t}{p}-\frac{q}{p}x\right)\right)\mathrm{d}x\right].$$

Proof. Similar to the proof of the previous theorem, we have

$$\begin{aligned}
T(C_{pn+q}^{s_1,\ldots,s_k,a_1n,\ldots,a_ln}) &= \frac{1}{pn+q}\prod_{j=1}^{pn+q-1}\left[2(k+l)-2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn+q}-2\sum_{i=1}^{l}\cos\frac{2\pi a_i nj}{pn+q}\right]\\
&= \frac{1}{pn+q}\prod_{t=0}^{p-1}\left(\prod_{\substack{j=1\\(j \bmod p)=t}}^{pn+q-1}\left[2(k+l)-2\sum_{i=1}^{k}\cos 2\pi s_i\frac{j}{pn+q}-2\sum_{i=1}^{l}\cos\frac{2\pi a_i nj}{pn+q}\right]\right).
\end{aligned}$$

Now let $j = pj' + t$ for some j' and t with $0 \le t \le p-1$. Then $1 \le j' \le n$ and

$$\frac{j'}{n+\frac{q}{p}} \le \frac{j}{pn+q} = \frac{pj'+t}{pn+q} = \frac{j'+\frac{t}{p}}{n+\frac{q}{p}} < \frac{j'+1}{n+\frac{q}{p}},$$

and $\cos 2\pi s_i \frac{j}{pn+q} = \cos 2\pi s_i \frac{j'+\frac{t}{p}}{n+\frac{q}{p}}$. Since $\frac{j}{p} = j' + \frac{t}{p}$ we have that

$$\begin{aligned}
\cos\frac{2\pi a_i nj}{pn+q} &= \cos\frac{2\pi a_i j}{p}\left(1-\frac{q}{pn+q}\right) \\
&= \cos\frac{2\pi a_i t}{p}\cos\frac{2\pi a_i q}{p}\frac{j}{pn+q} + \sin\frac{2\pi a_i t}{p}\sin\frac{2\pi a_i q}{p}\frac{j}{pn+q} \\
&= \cos\frac{2\pi a_i t}{p}\cos\frac{2\pi a_i q}{p}\frac{j'+\frac{t}{p}}{n+\frac{q}{p}} + \sin\frac{2\pi a_i t}{p}\sin\frac{2\pi a_i q}{p}\frac{j'+\frac{t}{p}}{n+\frac{q}{p}} \\
&= \cos 2\pi a_i\left(\frac{t}{p} - \frac{q}{p}\frac{j'+\frac{t}{p}}{n+\frac{q}{p}}\right).
\end{aligned}$$

Plugging these identities into the above formula and replacing each index j with the appropriate j' in the last expression, and then taking limits and simplifying the expression as in the proofs of Lemma 2 and Theorem 3, we obtain the following

$$\begin{aligned}
&\lim_{n\to\infty} T(C_{pn+q}^{s_1,\ldots,s_k,a_1n,\ldots,a_ln})^{\frac{1}{n}} \\
&= \exp\left[\lim_{n\to\infty}\ln\left(\prod_{j=1}^{pn+q-1}\left[2(k+l) - 2\sum_{i=1}^{k}\cos 2\pi s_i\frac{j}{pn+q} - 2\sum_{i=1}^{l}\cos\frac{2\pi a_i nj}{pn+q}\right]\right)\times\frac{1}{n+\frac{q}{p}}\times\frac{n+\frac{q}{p}}{n}\right] \\
&= 4^p\exp\left[\sum_{t=0}^{p-1}\int_0^1 \ln\left(2(k+l) - \sum_{i=1}^{k}\cos 2\pi s_k x - \sum_{i=1}^{l}\cos 2\pi a_i\left(\frac{t}{p}-\frac{q}{p}x\right)\right)\mathrm{d}x\right] \\
&= 4^p\exp\left[\sum_{t=0}^{p-1}\int_0^1 \ln\left(\sum_{i=1}^{k}\sin^2 \pi s_k x + \sum_{i=1}^{l}\sin^2 \pi a_i\left(\frac{t}{p}-\frac{q}{p}x\right)\right)\mathrm{d}x\right].
\end{aligned}$$

We can generalize even more to allow the jumps to be shifted slightly from linear:

Corollary 2. *Let $1 \le s_1 \le \cdots \le s_k$ and $0 \le a_1 \le \cdots \le a_l < p$ be integers and q the integer such that $0 < q < p$. Furthermore let $b_1, b_2, \ldots, b_l$ be any arbitrary integers. Then*

$$\lim_{n\to\infty} T\left(C_{pn+q}^{s_1,\ldots,s_k,a_1n+b_1,\ldots,a_ln+b_l}\right)^{\frac{1}{n}} = 4^p\exp\left[\sum_{t=0}^{p-1}\int_0^1 \ln\left(\sum_{i=1}^{k}\sin^2\pi s_k x + \sum_{i=1}^{l}\sin^2\pi\left[a_i\left(\frac{t}{p}-\frac{q}{p}x\right)+b_i x\right]\right)\mathrm{d}x\right].$$

Proof. Note that

$$\begin{aligned}
T\left(C_{pn+q}^{s_1,\ldots,s_k,a_1n+b_1,\ldots,a_ln+b_l}\right) &= \frac{1}{pn+q}\prod_{j=1}^{pn+q-1}\left[2(k+l) - 2\sum_{i=1}^{k}\cos\frac{2\pi s_i j}{pn+q} - 2\sum_{i=1}^{l}\cos\frac{2\pi(a_i n+b_i)j}{pn+q}\right] \\
&= \frac{1}{pn+q}\prod_{t=0}^{p-1}\left(\prod_{\substack{j=1\\(j \bmod p)=t}}^{pn+q-1}\left[2(k+l) - 2\sum_{i=1}^{k}\cos 2\pi s_i\frac{j}{pn+q} - 2\sum_{i=1}^{l}\cos\frac{2\pi(a_i n+b_i)j}{pn+q}\right]\right).
\end{aligned}$$

Same as before we may let $j = pj' + t$ for some j' and t with $0 \le t \le p-1$. Using the fact that $\frac{j}{pn+q} = \frac{j'+\frac{t}{p}}{n+\frac{q}{p}}$, simple manipulation gives

$$\begin{aligned}
\cos\frac{2\pi(a_i n+b_i)j}{pn+q} &= \cos\frac{2\pi a_i nj}{pn+q}\cos\frac{2\pi b_i j}{pn+q} - \sin\frac{2\pi a_i nj}{pn+q}\sin\frac{2\pi b_i j}{pn+q} \\
&= \cos 2\pi a_i\left(\frac{t}{p}-\frac{q}{p}\frac{j'+\frac{t}{p}}{n+\frac{q}{p}}\right)\cos\frac{2\pi b_i j}{pn+q} - \sin 2\pi a_i\left(\frac{t}{p}-\frac{q}{p}\frac{j'+\frac{t}{p}}{n+\frac{q}{p}}\right)\sin\frac{2\pi b_i j}{pn+q}
\end{aligned}$$

$$= \cos 2\pi a_i \left(\frac{t}{p} - \frac{q}{p}\frac{j' + \frac{t}{p}}{n + \frac{q}{p}}\right) \cos 2\pi b_i \frac{j' + \frac{t}{p}}{n + \frac{q}{p}} - \sin 2\pi a_i \left(\frac{t}{p} - \frac{q}{p}\frac{j' + \frac{t}{p}}{n + \frac{q}{p}}\right) \sin 2\pi b_i \frac{j' + \frac{t}{p}}{n + \frac{q}{p}}$$
$$= \cos 2\pi \left[a_i \left(\frac{t}{p} - \frac{q}{p}\frac{j' + \frac{t}{p}}{n + \frac{q}{p}}\right) + b_i \frac{j' + \frac{t}{p}}{n + \frac{q}{p}}\right].$$

We now (almost) copy the proof of the previous corollary to get

$$\lim_{n\to\infty} T\left(C_{pn+q}^{s_1,\ldots,s_k,a_1n+b_1,\ldots,a_ln+b_l}\right)^{\frac{1}{n}} = 4^p \exp\left[\sum_{t=0}^{p-1}\int_0^1 \ln\left(\sum_{i=1}^{k}\sin^2 \pi s_k x + \sum_{i=1}^{l}\sin^2 \pi \left[a_i\left(\frac{t}{p} - \frac{q}{p}x\right) + b_i x\right]\right) \mathrm{d}x\right]. \quad \square$$

A simple example. Recall the graph $C_{2n}^{1,n}$ with $2n$ vertices $0, 1, \ldots, 2n-1$ in which each node i is connected by one edge to node $(i+1) \bmod (2n)$, by one edge to node $(i-1) \bmod (2n)$, and by two edges to node $(i+n) \bmod (2n)$. It is known (e.g. Theorem 4 in [19]) that

$$T(C_{2n}^{1,n}) = \frac{n}{2}[(\sqrt{2}+1)^n + (\sqrt{2}-1)^n]^2, \tag{5}$$

so

$$\lim_{n\to\infty} T(C_{2n}^{1,n})^{\frac{1}{n}} = (\sqrt{2}+1)^2. \tag{6}$$

We now see that Theorem 3 yields exactly the same result. Theorem 3 immediately yields the closed form

$$\lim_{n\to\infty} T(C_{2n}^{1,n})^{\frac{1}{n}} = 4^2 \exp\left(\int_0^1 (\ln(\sin^2 \pi x + 1) + \ln(\sin^2 \pi x))\mathrm{d}x\right). \tag{7}$$

This integral can be evaluated by splitting into two parts.

To evaluate the first part we note that since $\sin^2 \pi x + 1 = 2 - \cos^2 \pi x$,

$$\int_0^1 \ln\ (\sin^2 \pi x + 1)\mathrm{d}x = \lim_{n\to\infty}\left(\sum_{j=1}^{n-1}\ln\left(2 - \cos^2\frac{\pi j}{n}\right)\right) \times \frac{1}{n} \tag{8}$$
$$= \lim_{n\to\infty} \ln\left(\prod_{j=1}^{n-1}\left(2 - \cos^2\frac{\pi j}{n}\right)\right) \times \frac{1}{n}.$$

We now recall the fact that $U_n(x)$, the nth Chebyshev polynomial of the second kind, satisfies ([4] and formulas (6) and (9) in [19])

$$U_{n-1}^2(x) = 4^{n-1}\prod_{j=1}^{n-1}\left(x^2 - \cos^2\frac{\pi j}{n}\right)$$

and

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x + \sqrt{x^2-1}\right)^{n+1} + \left(x - \sqrt{x^2-1}\right)^{n+1}\right].$$

Plugging back into (8) yields

$$\int_0^1 \ln\ (\sin^2 \pi x + 1)\mathrm{d}x = \lim_{n\to\infty} \ln\left(\frac{U_{n-1}^2(\sqrt{2})}{4^{n-1}}\right) \times \frac{1}{n}$$
$$= \ln\frac{(\sqrt{2}+1)^2}{4}.$$

To evaluate the second part, recall the well known identity, c.f., [1] 4.3.145,

$$\int_0^{\pi/2} \ln \sin t\mathrm{d}t = -\frac{\pi}{2}\ln 2.$$

Thus

$$\int_0^1 \ln(\sin^2 \pi x)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} \ln(\sin^2 u)\mathrm{d}u$$
$$= \frac{4}{\pi}\int_0^{\pi/2} \ln \sin u\mathrm{d}u = -\ln 4.$$

Plugging the values of the two integrals just evaluated into (7) gives

$$\lim_{n\to\infty} T(C_{2n}^{1,n})^{\frac{1}{n}} = 4^2 \exp\left[\ln \frac{(\sqrt{2}+1)^2}{4} - \ln 4\right] = (\sqrt{2}+1)^2,$$

which is exactly (6).

We now return to prove our main theorem:

Theorem 4. *Let $1 < d_1 \le \cdots \le d_l$ be fixed positive integers and $p = lcm(d_1, d_2, \ldots, d_l)$. Let $1 \le s_1 \le \cdots \le s_k$ be positive integers and $e_1, e_2, \ldots, e_l$ be arbitrary integers. Set $a_i = \frac{p}{d_i}$, and for $0 \le q < p$ also set $b_{q,i} = \left\lfloor \frac{q}{d_i} \right\rfloor + e_i$ Then, for fixed q,*

$$\lim_{\substack{m\to\infty \\ m \bmod p=q}} T\left(C_m^{s_1,s_2,\ldots,s_k,\left\lfloor \frac{m}{d_1}\right\rfloor+e_1,\left\lfloor \frac{m}{d_2}\right\rfloor+e_2,\ldots,\left\lfloor \frac{m}{d_l}\right\rfloor+e_l}\right)^{\frac{1}{m}}$$
$$= 4\exp\left(\frac{1}{p}\sum_{t=0}^{p-1}\int_0^1 \ln\left(\sum_{i=1}^k \sin^2 \pi s_k x + \sum_{i=1}^l \sin^2 \pi \left[a_i\left(\frac{t}{p} - \frac{q}{p}x\right) + b_{q,i}x\right]\right) dx\right).$$

Proof. The definitions of a_i, b_i imply

$$\text{if } m = pn+q \text{ then } \left\lfloor \frac{m}{d_i}\right\rfloor + e_i = a_i n + b_{q,i}. \tag{9}$$

Combining Theorem 3, Corollaries 1 and 2 yields, for fixed q,

$$\lim_{n\to\infty} T\left(C_{pn+q}^{s_1,\ldots,s_k,a_1n+b_{q,1},\ldots,a_ln+b_{q,l}}\right)^{\frac{1}{n}}$$
$$= 4^p \exp\left(\sum_{t=0}^{p-1}\int_0^1 \ln\left(\sum_{i=1}^k \sin^2 \pi s_k x + \sum_{i=1}^l \sin^2 \pi \left[a_i\left(\frac{t}{p} - \frac{q}{p}x\right) + b_{q,i}x\right]\right) dx\right).$$

This implies

$$\lim_{\substack{m\to\infty \\ m \bmod p=q}} T\left(C_m^{s_1,s_2,\ldots,s_k,\left\lfloor \frac{m}{d_1}\right\rfloor+e_1,\left\lfloor \frac{m}{d_2}\right\rfloor+e_2,\ldots,\left\lfloor \frac{m}{d_l}\right\rfloor+e_l}\right)^{\frac{1}{m}}$$
$$= \lim_{n\to\infty} T\left(C_{pn+q}^{s_1,\ldots,s_k,a_1n+b_{q,1},\ldots,a_ln+b_{q,l}}\right)^{\frac{1}{pn+q}}$$
$$= 4\exp\left(\frac{1}{p}\sum_{t=0}^{p-1}\int_0^1 \ln\left(\sum_{i=1}^k \sin^2 \pi s_k x + \sum_{i=1}^l \sin^2 \pi \left[a_i\left(\frac{t}{p} - \frac{q}{p}x\right) + b_{q,i}x\right]\right) dx\right). \quad \square$$

3. The asymptotics of $T(C_n^{s_1,s_2,\ldots,s_k})^{\frac{1}{n}}$

In the previous section, we saw that, if $s_1, s_2, \ldots, s_k$ are fixed, then $T(C_n^{s_1,s_2,\ldots,s_k})^{\frac{1}{n}}$ converges to a constant dependent upon the s_i. In this section we discuss how this constant changes as n and the jumps s_i themselves tend to infinity.

Lemma 5. *If $s_1, s_2, \ldots, s_k$ are arbitrary positive integers and $t \le k$, then*

$$\lim_{s_1,s_2,\ldots,s_t\to\infty}\lim_{n\to\infty} T\left(C_n^{s_1,s_2,\ldots,s_k}\right)^{\frac{1}{n}} = 4\exp\left(\int_0^1\cdots\int_0^1 \ln\left(\sum_{i=1}^t \sin^2 \pi x_i + \sum_{i=t+1}^k \sin^2 \pi s_i x\right) dx_1\cdots dx_t dx\right)$$

where $s_1, s_2, \ldots, s_t \to \infty$ in any arbitrary order.

Proof. From Lemma 2, we have that, if $s_1, s_2, \ldots, s_k$ are fixed, then

$$\lim_{n\to\infty} \frac{\ln\left(T\left(C_n^{s_1,s_2,\ldots,s_k}\right)\right)}{n} = \ln 4 + \int_0^1 \ln\left(\sin^2 \pi s_1 x + \sin^2 \pi s_2 x + \cdots + \sin^2 \pi s_k x\right) dx.$$

Setting $s_1x = x_1$ yields

$$\lim_{n\to\infty} \frac{\ln(T(C_n^{s_1,s_2,\ldots,s_k}))}{n} = \ln 4 + \int_0^{s_1} \ln\left(\sin^2 \pi x_1 + \sin^2 \pi s_2 \frac{x_1}{s_1} + \cdots + \sin^2 \pi s_k \frac{x_1}{s_1}\right) \frac{1}{s_1} \mathrm{d}x_1$$
$$= \ln 4 + \sum_{j=0}^{s_1-1} \left(\int_j^{j+1} \ln\left(\sin^2 \pi x_1 + \sin^2 \pi s_2 \frac{x_1}{s_1} + \cdots + \sin^2 \pi s_k \frac{x_1}{s_1}\right) \mathrm{d}x_1\right) \frac{1}{s_1}.$$

For all s_1, j and x_1 that appear in the last integral, $\frac{j}{s_1} \le \frac{x_1}{s_1} \le \frac{j+1}{s_1}$. Furthermore, $\sin^2 \pi(t+j) = \sin^2 \pi t$. Therefore, fixing $s_2, s_3, \ldots, s_k$ and writing $x_1 = t + j$ for $x_1 \in [j, j+1)$, permits evaluating

$$\lim_{s_1\to\infty} \sum_{j=0}^{s_1-1} \left(\int_j^{j+1} \ln\left(\sin^2 \pi x_1 + \sum_{i=2}^{k} \sin^2 \pi s_i \frac{x_1}{s_1}\right) \mathrm{d}x_1\right) \frac{1}{s_1} = \lim_{s_1\to\infty} \sum_{j=0}^{s_1-1} \left(\int_0^{1} \ln\left(\sin^2 \pi t + \sum_{i=2}^{k} \sin^2 \pi s_i \frac{t+j}{s_1}\right) \mathrm{d}t\right) \frac{1}{s_1}$$
$$= \int_0^1 \int_0^1 \ln\left(\sin^2 \pi x_1 + \sum_{i=2}^{k} \sin^2 \pi s_i y\right) \mathrm{d}x_1 \mathrm{d}y.$$

Thus

$$\lim_{s_1\to\infty} \lim_{n\to\infty} \frac{\ln(T(C_n^{s_1,s_2,\ldots,s_k}))}{n} = \ln 4 + \lim_{s_1\to\infty} \sum_{j=0}^{s_1-1} \left(\int_0^{1} \ln\left(\sin^2 \pi x_1 + \sin^2 \pi s_2 \frac{x_1}{s_1} + \cdots + \sin^2 \pi s_k \frac{x_1}{s_1}\right) \mathrm{d}x_1\right) \frac{1}{s_1}$$
$$= \ln 4 + \int_0^1 \int_0^1 \ln\left(\sin^2 \pi x_1 + \sin^2 \pi s_2 y + \cdots + \sin^2 \pi s_k y\right) \mathrm{d}x_1 \mathrm{d}y.$$

Setting $s_2 y = x_2$ similarly yields

$$\lim_{s_1\to\infty} \lim_{n\to\infty} \frac{\ln(T(C_n^{s_1,s_2,\ldots,s_k}))}{n} = \ln 4 + \int_0^{s_2} \int_0^1 \ln\left(\sin^2 \pi x_1 + \sin^2 \pi x_2 + \sin^2 \pi s_3 \frac{x_2}{s_2} + \cdots + \sin^2 \pi s_k \frac{x_2}{s_2}\right) \frac{1}{s_2} \mathrm{d}x_1 \mathrm{d}x_2$$
$$= \ln 4 + \sum_{j=0}^{s_2-1} \left(\int_j^{j+1} \int_0^1 \ln\left(\sin^2 \pi x_1 + \sin^2 \pi x_2 + \sin^2 \pi s_3 \frac{x_2}{s_2} + \cdots + \sin^2 \pi s_k \frac{x_2}{s_2}\right) \mathrm{d}x_1 \mathrm{d}x_2\right) \frac{1}{s_2}.$$

Again, for all s_2, j and x_2 that appear in the integral, $\frac{j}{s_2} \le \frac{x_2}{s_2} \le \frac{j+1}{s_2}$. Therefore, for fixed $s_3, s_4, \ldots, s_k$, the same reasoning as above yields

$$\lim_{s_2\to\infty} \lim_{s_1\to\infty} \lim_{n\to\infty} \frac{\ln(T(C_n^{s_1,s_2,\ldots,s_k}))}{n} = \ln 4 + \int_0^1 \int_0^1 \int_0^1 \ln\left(\sin^2 \pi x_1 + \sin^2 \pi x_2 + \sin^2 \pi s_3 y + \cdots + \sin^2 \pi s_k y\right) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y.$$

Assume for the moment that $t < k$. Continuing in the same fashion for $s_3, \ldots, s_t$ gives,

$$\lim_{s_t\to\infty} \lim_{s_{t-1}\to\infty} \cdots \lim_{s_1\to\infty} \lim_{n\to\infty} \frac{\ln(T(C_n^{s_1,s_2,\ldots,s_k}))}{n}$$
$$= \ln 4 + \int_0^1 \int_0^1 \cdots \int_0^1 \ln\left(\sum_{i=1}^{t} \sin^2 \pi x_i + \sum_{i=t+1}^{k} \sin^2 \pi s_i x\right) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_t \mathrm{d}x$$

which is equivalent to the statement of the lemma.

For the case $t = k$ note that the analysis of $t = k - 1$ gives, for fixed s_k,

$$\lim_{s_{k-1}\to\infty} \lim_{s_{k-2}\to\infty} \cdots \lim_{s_1\to\infty} \lim_{n\to\infty} \frac{\ln(T(C_n^{s_1,s_2,\ldots,s_k}))}{n}$$
$$= \ln 4 + \int_0^1 \int_0^1 \cdots \int_0^1 \ln\left(\sum_{i=1}^{k-1} \sin^2 \pi x_i + \sin^2 \pi s_k x\right) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_{k-1} \mathrm{d}x$$
$$= \ln 4 + \sum_{j=0}^{s_k-1} \left(\int_j^{j+1} \int_0^1 \cdots \int_0^1 \ln\left(\sum_{i=1}^{k-1} \sin^2 \pi x_i + \sin^2 \pi x_k\right) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_k\right) \frac{1}{s_k}.$$

Taking the limit as $s_k \to \infty$ gives

$$\lim_{s_k\to\infty}\lim_{s_{k-1}\to\infty}\cdots\lim_{s_1\to\infty}\lim_{n\to\infty}\frac{\ln(T(C_n^{s_1,s_2,\ldots,s_k}))}{n}$$
$$= \ln 4 + \int_0^1\int_0^1\cdots\int_0^1 \ln\left(\sum_{i=1}^k \sin^2\pi x_i\right)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_k.$$

To conclude, note that the proof as given requires that the limits be taken in the specific order

$$\lim_{s_t\to\infty}\lim_{s_{t-1}\to\infty}\cdots\lim_{s_1\to\infty}.$$

The fact that the order in which the limits is taken does not matter, i.e., that the $s_1,\ldots,s_t\to\infty$ in any arbitrary way, follows from the symmetry of $T(C_n^{s_1,s_2,\ldots,s_k})$ with respect to $s_1, s_2, \ldots, s_k$. □

We restate the special case $t = k$ as a theorem:

Theorem 6. *Let $s_1, s_2, \ldots, s_k$ be arbitrary positive integers. Then*

$$\lim_{s_1,s_2,\ldots,s_k\to\infty}\lim_{n\to\infty} T(C_n^{s_1,s_2,\ldots,s_k})^{\frac{1}{n}} = 4\exp\left(\int_0^1\int_0^1\cdots\int_0^1 \ln\left(\sum_{i=1}^k \sin^2\pi x_i\right)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_k\right).$$

Interestingly, this quantity is exactly the asymptotic limit of the number of spanning trees in k-dimensional square tori as derived by Garcia, Noy and Tejel in [9]. Let T_n^k be the number of spanning trees in the k-dimensional square torus with n-vertices, i.e., each dimension has span $n^{1/k}$. In [9] it is shown that $\lim_{n\to\infty}\left(T_n^k\right)^{1/n}$ is exactly the quantity given in Theorem 6.

For another special case let $G(m, n)$ and $TS(m, n)$ denote, respectively, the 2-dimensional $m\times n$ grid and torus. [6,9] tell us that

$$\begin{aligned}\lim_{m,n\to\infty} T(TS(m,n))^{\frac{1}{mn}} &= \lim_{m,n\to\infty} T(G(m,n))^{\frac{1}{mn}}\\ &= 4\exp\left(\int_0^1\int_0^1 \ln\left(\sin^2\pi x+\sin^2\pi y\right)\mathrm{d}x\mathrm{d}y\right)\\ &= 3.20991230\ldots.\end{aligned}$$

As noted in [10], when drawing the circulant graph $C_{mn}^{1,n}$ on the grid $G(n, m)$ (mapping node k in $C_{mn}^{1,n}$ to the unique node (i, j) in $G(n, m)$ where $k = ni + j$), $C_{mn}^{1,n}$ is actually identical to the torus $TS(n, m)$ except for side edges all of whose left endpoints are shifted up by one. Thus, in [10] it was conjectured that the asymptotics of the circulant would be similar to that of the torus. This will now be seen to be true:

Corollary 3.

$$\lim_{m\to\infty}\lim_{n\to\infty} T(C_{mn}^{1,m})^{\frac{1}{mn}} = \lim_{m,n\to\infty} T(TS(m,n))^{\frac{1}{mn}} = \lim_{m,n\to\infty} T(G(m,n))^{\frac{1}{mn}}.$$

Proof. Lemma 5 with $n = s$, $s_1 = m$ and $t = 1$ gives

$$\lim_{m\to\infty}\lim_{s\to\infty} T\left(C_s^{1,m}\right)^{1/s} = 4\exp\left(\int_0^1\int_0^1 \ln\left(\sin^2\pi x+\sin^2\pi y\right)\mathrm{d}x\mathrm{d}y\right).$$

Thus

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty} T(C_{mn}^{1,m})^{\frac{1}{mn}} &= \lim_{m\to\infty}\lim_{s\to\infty} T(C_s^{1,m})^{\frac{1}{s}}\\ &= 4\exp\left(\int_0^1\int_0^1 \ln\left(\sin^2\pi x+\sin^2\pi y\right)\mathrm{d}x\mathrm{d}y\right). \quad\square\end{aligned}$$

Lemma 5 states that, in calculating the limits, as long as $n\to\infty$ first, the order in which the s_i go to infinity does not matter. An interesting remaining question here would be to show for the circulants that, as in the case of k-dimensional grids and tori, the *order* in which the limit over n is taken does not matter either. That is, viewing $T(C_n^{s_1,s_2,\ldots,s_k})$ as a function of $n, s_1, s_2, \ldots, s_k$, is it true that

$$\lim_{n,s_1,s_2,\ldots,s_k\to\infty} T(C_n^{s_1,s_2,\ldots,s_k})^{\frac{1}{n}} = 4\exp\left(\int_0^1\cdots\int_0^1 \ln\left(\sin^2\pi x_1+\cdots+\sin^2\pi x_k\right)\mathrm{d}x_1\cdots\mathrm{d}x_k\right).$$

4. Concluding remarks

In this paper we derived closed formulas for $\lim_{n\to\infty}(T(C(n)))^{\frac{1}{n}}$, where $C_{(n)}$ is a circulant graph with given fixed and linear jumps and $T(C(n))$ denotes the number of spanning trees in $C(n)$. As a secondary result, our techniques can also be used to derive the limiting asymptotics of the number of spanning trees of circulant graphs $C_n^{s_1,s_2,\ldots,s_k}$ when jumps $s_1, s_2, \ldots, s_k$ and n tend to infinity. We saw that this is exactly the same as the limiting asymptotics of the number of spanning trees in the k-dimensional tori (and k-dimensional grids) when the number of vertices in the tori (and grids) tend to infinity.

We conclude with a question about growth rates. In Lemma 2 we showed that for any fixed integers $1 \le s_1 \le s_2 \le \cdots \le s_k$,

$$\lim_{n\to\infty} T(C_n^{s_1,s_2,\ldots,s_k})^{\frac{1}{n}} = \lim_{n\to\infty} \frac{T(C_{n+1}^{s_1,s_2,\ldots,s_k})}{T(C_n^{s_1,s_2,\ldots,s_k})}.$$

Note that this quantity represents the average growth rate of the number of spanning trees of the circulant graph $C_n^{s_1,s_2,\ldots,s_k}$. Since Lemma 2 also tells us that this value is equal to

$$4\int_0^1 \ln\left(\sum_{i=1}^{k} \sin^2 \pi s_i x\right) \mathrm{d}x, \tag{10}$$

finding the jumps s_i that maximize or minimize the average growth rate among all families of $2k$-regular circulant graphs would be equivalent to finding $s_1, s_2, \ldots, s_k$ that maximize or minimize (10). To the best of our knowledge, this problem has only been addressed for directed circulant graphs with $k = 2$ [14] (with only partial solutions). It would be interesting to try and solve this more generally.

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References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, U.S. National Bureau of Standards, Washington D.C, 1964.
[2] S.D. Bedrosian, Tree counting polynomials for labelled graphs, J. Franklin Inst. 312 (1981) 417–430.
[3] N. Biggs, Algebraic Graph Theory, Second ed., Cambridge University Press, London, 1993.
[4] F.T. Boesch, H. Prodinger, Spanning tree formulas and Chebyshev polynomials, Graphs Combin. 2 (1986) 191–200.
[5] F.T. Boesch, J.F. Wang, A Conjecture on the number of spanning trees in the square of a cycle, in: Notes from New York Graph Theory Day V, 1982, page 16.
[6] X. Chen, An asymptotic enumeration theorem for the number of spanning trees in grids and tori (in Chinese), J. Zhangzhou Teach. Coll. Nat. Sci. 14 (2) (2001) 7–12.
[7] X. Chen, The number of spanning trees in directed circulant graphs with non-fixed jumps, Discrete Math. 307 (2007) 1873–1880.
[8] D. Cvetkovič, M. Doob, H. Sachs, Spectra of Graphs: Theory and Applications, Third ed., Johann Ambrosius Barth, Heidelberg, 1995.
[9] A. Garcia, M. Noy, J. Tejel, The asymptotic number of spanning trees in d-dimensional square lattices, J. Combin. Math. Combin. Comput. 44 (2003) 109–113.
[10] M. Golin, Y. Leung, Y. Wang, X. Yong, Counting structures in grid graphs, cylinders and tori using transfer matrices: Survey and new results, in: The Proceedings of SIAM ALENEX/ANALCO (Workshop on Analytic Algorithms and Combinatorics), British Columbia, Canada, 2005.
[11] M.J. Golin, Y.C. Leung, Y.J. Wang, Counting spanning trees and other structures in non-constant jump circulant graphs, in: The 15th Annual International Symposium on Algorithms and Computation, 2004, pp. 508–521.
[12] G. Kirchhoff, Über die Auflösung der gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird, Ann. Phys. Chem. 72 (1847) 497–508.
[13] Z. Lonc, K. Parol, J.M. Wojciechowski, On the asymptotic behavior of the maximum number of spanning trees in circulant graphs, Networks 30 (1) (1997) 47–56.
[14] Z. Lonc, K. Parol, J.M. Wojciechowski, On the number of spanning trees in directed circulant graphs, Networks 37 (3) (2001) 129–133.
[15] R. Lyons, Asymptotic enumeration of spanning trees, Combin. Probab. Comput. 14 (4) (2005) 491–522.
[16] X. Yong, Talip, Acenjian, The numbers of spanning trees of the cubic cycle C_N^3 and the quadruple cycle C_N^4, Discrete Math. 169 (1997) 293–298.
[17] F. Zhang, X. Yong, Asymptotic enumeration theorems for the numbers of spanning trees and Eulerian trails in circulant digraphs & graphs, Sci. China Ser. A 43 (3) (1999) 264–271.
[18] Y. Zhang, X. Yong, M. Golin, The number of spanning trees in circulant graphs, Discrete Math. 223 (2000) 337–350.
[19] Y. Zhang, X. Yong, M. Golin, Chebyshev polynomials and spanning tree formulas for circulant and related graphs, Discrete Math. 298 (2005) 334–364.