Revista Integración
Escuela de Matemáticas
Universidad Industrial de Santander
Vol. 31, No. 1, 2013, pág. 1–14

An extension of the $I + S_{max}$ preconditioner for the Gauss-Seidel method

Isnardo Arenas*, Paul Castillo, Xuerong Yong

University of Puerto Rico, Department of Mathematical Sciences, Mayagüez, Puerto Rico, 00681, US.

Abstract. A preconditioning technique based on the application of a fixed but arbitrary number of $I + S_{max}$ steps is proposed. A reduction of the spectral radius of the Gauss-Seidel iteration matrix is theoretically analyzed for diagonally dominant Z-matrices. In particular, it is shown that after a finite number of steps this matrix reduces to null matrix. To illustrate the performance of the proposed technique numerical experiments on a wide variety of matrices are presented. Point and block versions of the preconditioner are numerically studied.

Keywords: Preconditioning, Gauss-Seidel method, regular splitting, point and block preconditioners.
MSC2010: XX00YY, AA99BB

Una extensión del precondicionador $I + S_{max}$ para el método de Gauss-Seidel

Resumen. Se propone una técnica de precondicionamiento para el método de Gauss-Seidel basada en la aplicación de una cantidad de pasos arbitrarios pero fijos del precondicionador $I + S_{max}$. Se analiza de manera teórica la reducción del radio espectral de la matriz de iteración del método de Gauss-Seidel para Z-matrices diagonalmente dominantes. En particular, se demuestra que después de un número finito de pasos esta matriz se reduce a una matriz nula. Para ilustrar la eficacia de la técnica propuesta se presentan experimentos numéricos para una amplia variedad de matrices. Se estudian numéricamente versiones puntuales y de bloques del precondicionador.

Palabras claves: Precondicionamiento, método Gauss-Seidel, descomposiciones regulares, precondicionadores de punto y bloque.

*Corresponding author: *E-mail*: isnardo.arenas@upr.edu.
Received: 03 December 2012, Accepted: 23 March 2013.

1. Introduction

Preconditioning techniques for accelerating the convergence of classical relaxation methods such as Jacobi and Gauss-Seidel have been proposed for diagonally dominant Z-matrices in [8, 9, 10, 12, 13, 17]. Extensions of these preconditioners to other classes of matrices, such as M-matrices or H-matrices, and new preconditioners have been recently presented in [4, 7, 11, 14, 16, 19]. The rationale of these techniques can be described as follows. Consider the linear system $Ax = b$, where A is an $n \times n$ nonsingular matrix and its decomposition $A = -L + D - U$, where $-L$ and $-U$ are the strict lower and upper triangular parts of A and D its main diagonal. Gauss-Seidel iteration can be recast as a fixed point iteration of the form

$$x_{n+1} \;=\; M^{-1}Nx_n + M^{-1}b,$$

where $M = D - L$ and $N = U$. It is well known (see for example [18]) that this iteration converges if and only if $\rho\left(M^{-1}N\right) < 1$. Then a transformation of the original system into an equivalent system, $\widehat{A}x = \widehat{b}$, is sought such that $\rho(\widehat{M}^{-1}\widehat{N}) < \rho(M^{-1}N)$, where $\widehat{M}$ and $\widehat{N}$ are the decomposition matrices associated with the Gauss-Seidel iteration applied to $\widehat{A}x = \widehat{b}$. Typically the new system is obtained by multiplying the original one by a suitable preconditioning matrix, $\mathcal{P}$, thus $\widehat{A} = \mathcal{P}A$ and $\widehat{b} = \mathcal{P}b$.

The success of such preconditioning technique relies on the reduction of the spectral radius of the iteration matrix. A simple strategy consists of reducing the number of non zero entries in $M^{-1}N$. In [12] a selected column of the iteration matrix was eliminated. Another approach is to annihilate some entries in $-U$ therefore in N. In [13], the first upper co-diagonal was canceled using elementary row transformations. A theoretical analysis of this approach was first presented in [6], where it was shown that for a certain class of matrices, namely Z-matrices, this strategy improves the performance of the Gauss-Seidel iteration. Various generalizations of the above method have been proposed. A convergence analysis of these techniques was presented in [8, 10] for diagonally dominant Z-matrices and more recently extended to H-matrices in [11].

In [9], a preconditioner called $I+S_{max}$ was analyzed and compared with other techniques. Here entries in U with the largest absolute values on each row are annihilated. In this work we present and analyze an extension of that preconditioner. The rationale of our strategy is to reduce the matrix N to 0, after applying a fixed number of times the preconditioning step $I + S_{max}$. We show that after a finite number of steps the upper triangular part of the preconditioned matrix reduces to 0 when A is a diagonally dominant Z-matrix. This results shows an improvement of the proposed strategy compared to the classical $I + S_{max}$ preconditioner. Finally, in Section 4 we show the performance of the proposed preconditioner on a wide variety of matrices including matrices not covered by the theory.

2. Preliminary results

Now we review the $I + S_{max}$ preconditioner, introduced in [9], applied to a non-singular diagonally dominant Z-matrix with positive diagonal A. The goal is to cancel, on each row of the upper triangular part of A, the first entry $a_{i,j}$ with the largest absolute value. Let $S_{max} = [s_{i,j}]$ be the $n \times n$ matrix associated to A such that

$$s_{i,j} \;=\; \begin{cases} -a_{i,j}/a_{j,j}, & \text{if } j > i \text{ and } j = k_i, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where

$$k_i \;=\; \min\left\{ j \;:\; \max_{i<k} |a_{i,k}| = |a_{i,j}| \right\}.$$

The construction of matrix $\widehat{A} = (I + S_{max})A$ can be achieved by simple row manipulations. Observe that row $\widehat{A}_{i,*}$ can be written as follows:

$$\widehat{A}_{i,*} \;=\; A_{i,*} \;+\; s_{i,k_i} A_{k_i,*},$$

and can be computed independently of the other rows.

Lemma 2.1. *Let $A \geq 0$ be an $n \times n$ real matrix and $x \in \mathbb{R}^n$ is positive. If $\alpha \geq 0$ and $Ax \leq \alpha x$, then $\rho(A) \leq \alpha$; if A is irreducible and for a some $x \geq 0$ we have $Ax \leq \alpha x$ and $Ax \neq \alpha x$, then $\rho(A) < \alpha$.*

The proof of this lemma is straightforward by the Perron-Frobenius theorem (for example, a proof is given in [2]). Notice that if A is an $n \times n$ Z-matrix and $\mathcal{P} = (I + S_{\max})$, then $\mathcal{P}A$ is again an Z-matrix.

Lemma 2.2. *Let A be an $n \times n$ matrix and $D = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n)$ a nonsingular diagonal matrix. Then $\rho(M^{-1}N) = \rho(M_D^{-1}N_D)$ where $A = M - N$ and $DA = M_D - N_D$ are the regular Gauss-Seidel splittings.*

Proof. Note that for a diagonal matrix D, the regular splittings of DA are $M_D = DM$ and $N_D = DN$, so $\rho(M_D^{-1}N_D) = \rho((DM)^{-1}DN) = \rho(M^{-1}D^{-1}DN) = \rho(M^{-1}N)$. ☑

Observe that the claim of Lemma 2.2 is not true when D is not diagonal, because in that case $M_D \neq DM$.

Lemma 2.3. *If A is an $n \times n$ nonsingular diagonally dominant Z-matrix with positive diagonals, and $\mathcal{P} = (I + S_{max})$, then $\widehat{A} = \mathcal{P}A$ is again nonsingular diagonally dominant Z-matrix with positive diagonals and non-singular.*

Proof. Note that $(\widehat{A})_{i,j} = a_{i,j} - \frac{a_{i,k_i}}{a_{k_i,k_i}} a_{k_i,j}$, and $\left|\frac{a_{k_j,i}}{a_{k_i,k_i}}\right| \leq 1$. If $\left|\frac{a_{k_j,i}}{a_{k_i,k_i}}\right| < 1$; then,

$$(\widehat{A})_{i,i} = a_{i,i} - \frac{a_{i,k_i}}{a_{k_i,k_i}} a_{k_i,i} > 0.$$

If $\left|\frac{a_{k_j,i}}{a_{k_i,k_i}}\right| = 1$, by the definition of a diagonally dominant Z-matrix, $-a_{k_i,i} = a_{k_i,k_i}$ and $a_{k_i,j} = 0$ for $j \neq i$ and $j \neq k_i$. Now suppose that $(\widehat{A})_{i,i} = 0$; then $a_{i,i} = -a_{i,k_i}$ and $a_{i,j} = 0$ for $j \neq i$ and $j \neq k_i$. Then the rows i and k_i are linearly dependent, contradicting to the assumption that A is non-singular. Hence, all the diagonal entries

of $\widehat{A}$ are positive, and $a_{i,i} \geq \sum_{j=1\, j\neq i}^{n} |a_{i,j}|$, since A is an Z-matrix and $\sum_{j=1}^{n} a_{i,j} \geq 0$ for all i. Being A a diagonally dominant Z-matrix, we have

$$\begin{aligned}
\left|\widehat{A}_{i,i}\right| - \sum_{j\neq i}^{n} \left|\widehat{A}_{i,j}\right| &= a_{i,i} - \frac{a_{i,k_i}}{a_{k_i,k_i}} a_{k_i,i} - \sum_{j\neq i}^{n} -\widehat{A}_{i,j} \\
&= a_{i,i} - \frac{a_{i,k_i}}{a_{k_i,k_i}} a_{k_i,i} + \sum_{j=1\, j\neq i}^{n} a_{i,j} - \frac{a_{i,k_i}}{a_{k_i,k_i}} a_{k_i,j} \\
&= \sum_{j=1}^{n} a_{i,j} - \frac{a_{i,k_i}}{a_{k_i,k_i}} \sum_{j=1}^{n} a_{k_i,j} \\
&\geq 0. \qquad (2)
\end{aligned}$$

So, $\left|\widehat{A}_{i,i}\right| \geq \sum_{j=1\, j\neq i}^{n} \left|\widehat{A}_{i,j}\right|$. That is, $\widehat{A}$ is a diagonally dominant Z-matrix. If A is strictly diagonally dominant then (2) is a strict inequality. ☑

Lemma 2.4. *If A is an $n \times n$ M-matrix and $\mathcal{P} = (I + S_{max})$, then $\widehat{A}_h = \mathcal{P}A$ is also an M-matrix.*

Proof. It is well-known that

- C is an M-matrix if and only if there is a nonsingular diagonal matrix D such that CD is strictly diagonally dominant matrix by row.
- C is an M-matrix if and only if, for any nonsingular diagonal matrix D_x, CD_x is again an M-matrix.

So, applying the proof of Lemma 2.3 to the matrix CD and the above two facts we see readily that $\widehat{A}_h = \mathcal{P}A$ is again an M-matrix. ☑

Lemma 2.5. *Let A be an irreducible diagonally dominant M-matrix, and $A = M - N$ Gauss-Seidel splitting. Then the Gauss-Seidel iteration matrix $\widehat{M}^{-1}\widehat{N}$ associated with the preconditioned matrix $\widehat{A} = \mathcal{P}A = \widehat{M} - \widehat{N}$, $\mathcal{P} = (I + S_{max})$ satisfies*

$$\rho(\widehat{M}^{-1}\widehat{N}) \;<\; \rho(M^{-1}N) \;<\; 1. \qquad (3)$$

Proof. Since M^{-1} and N are nonnegative, and $\rho(M^{-1}N) < 1$, there exists a nonnegative vector x satisfying $M^{-1}Nx = \rho(M^{-1}N)x$, and so $Ax = (M - N)x = M(I - M^{-1}N)x = (1 - \rho(M^{-1}N))(A + N)x$. Solving the equation for Ax yields

$$Ax \;= \frac{1 - \rho(M^{-1}N)}{\rho(M^{-1}N)} Nx \geq 0.$$

Since $\widehat{M}^{-1} \geq 0$ and $\mathcal{P} = I + S_{max} \geq I(\neq)$, then

$$\widehat{M}^{-1}\mathcal{P} \geq \widehat{M}^{-1} \geq M^{-1}, \quad \widehat{M}^{-1}\mathcal{P} \neq \widehat{M}^{-1} \quad \text{and} \quad \widehat{M}^{-1} \neq M^{-1}.$$

By simple algebraic manipulations we have

$$0 \;\leq\; (\widehat{M}^{-1}\mathcal{P} - M^{-1})Ax \;=\; (\widehat{M}^{-1}\mathcal{P} - M^{-1})(M-N)x \;=\; M^{-1}Nx - \widehat{M}^{-1}\widehat{N}x.$$

Hence,

$$\widehat{M}^{-1}\widehat{N}x \;\leq\; \rho(M^{-1}N)x.$$

By Lemma 2.1, we obtain that $\rho(\widehat{M}^{-1}\widehat{N}) \leq \rho(M^{-1}N)$. Now, if

$$\rho(M^{-1}N) = \rho(\widehat{M}^{-1}\widehat{N})$$

then $\widehat{M}^{-1}\widehat{N}x \leq \rho(\widehat{M}^{-1}\widehat{N})x$, and so, by the Perron Frobenius Theorem, $\widehat{M}^{-1}\widehat{N}x = \rho(\widehat{M}^{-1}\widehat{N})x$, and so x must be the eigenvector of $\widehat{M}^{-1}\widehat{N}$ corresponding to $\rho(\widehat{M}^{-1}\widehat{N})$, which is impossible. This completes the proof. ☑

If we replace the hypothesis of being a Z-matrix by H-matrix in the previous lemma, the inequality (3) is not satisfied, but we still have

$$\rho(\widehat{M}^{-1}\widehat{N}) \;<1 \quad \text{and} \quad \rho(M^{-1}N) \;<1.$$

3. The $IpS_{max}(A,k)$ preconditioner

Since the action of $I + S_{max}$ on matrix A is to eliminate a single entry on each row of its upper triangular part, by repeating this process we expect more entries to be eliminated. The proposed preconditioner consists of applying the $I + S_{max}$ repeatedly in a fixed number of times. That is, let $A_0 = IpS_{max}(A,0) = A$; then we define A_k recursively as

$$A_k \;=\; IpS_{max}(A_{k-1},1) \;=\; (I + S_{max}^{k-1})A_{k-1}, \quad \forall k \geq 1, \tag{4}$$

where $I + S_{max}^{k-1}$ is the preconditioner defined in [9] with respect to matrix A_{k-1}.

We show the convergence properties for any irreducibly diagonally dominant Z-matrix with positive diagonals.

Theorem 3.1. *Let $A = (a_{i,j})$ be an $n \times n$ diagonally dominant matrix. Then there exists a matrix dependent on integer k_A, such that A_{k_A} is lower triangular.*

Proof. We proceed by induction. For $n = 2$, notice that if $a_{1,2} = 0$, $I + S_{max}^0$ reduces to I and $A_1 = A_0$ the matrix is lower triangular; otherwise we have

$$A_1 \;=\; (I + S_{max}^0)A_0 \;=\; \begin{pmatrix} 1 & -\frac{a_{1,2}}{a_{2,2}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \;=\; \begin{pmatrix} a_{1,1} - \frac{a_{1,2}a_{2,1}}{a_{2,2}} & 0 \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

That completes the proof for $n = 2$.

Now, suppose that A is a matrix of size $(k+1) \times (k+1)$ satisfying the above hypothesis and assume that the result holds for any $k \times k$ diagonally dominant matrix. We partition matrix A according to the first row and column as follows:

$$A \;=\; \left(\begin{array}{c|ccc} a_{1,1} & a_{1,2} & \cdots & a_{1,k+1} \\ \hline a_{2,1} & & & \\ \vdots & & A_{2,2} & \\ a_{k+1,1} & & & \end{array}\right).$$

Since $A_{2,2}$ is a $k \times k$ diagonally dominant matrix, by induction hypothesis there exists $k_1 \in \mathbb{N}$ such that

$$A_{k_1} = \left(\begin{array}{c|ccccc} * & * & * & * & \dots & * \\ \hline * & * & 0 & \dots & \dots & 0 \\ * & * & * & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ * & * & \dots & * & * & 0 \\ * & * & \dots & * & * & * \end{array}\right).$$

We now consider the partition according to the $k \times k$ leading sub-matrix of A_{k_1}, that is

$$A_{k_1} = \left(\begin{array}{ccccc|c} * & * & * & * & \dots & * \\ * & * & 0 & \dots & \dots & 0 \\ * & * & * & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ * & * & \dots & * & * & 0 \\ \hline * & * & \dots & * & * & * \end{array}\right).$$

Case 1: $a_{1,k+1}^{k_1} = 0$. Since the $k \times k$ leading sub-matrix of A_{k_1} satisfies the properties of the theorem, we can apply the induction hypothesis to that block; then there exists $k_2 \in \mathbb{N}$ such that the leading sub-matrix can be reduced to a lower triangular matrix, hence $A_{k_1+k_2}$ is lower triangular. The theorem holds in this case with $n_A = k_1 + k_2$.

Case 2: $a_{1,k+1}^{k_1} \neq 0$. Then two possibilities must be considered: the first is when $\exists r \leq k_2$, $a_{1,k+1}^{k_1+r} = 0$, that is, we fall in Case 1 after $k_1 + r$ preconditioning steps; and finally, if $\forall r \leq k_2$, $a_{1,k+1}^{k_1+r} \neq 0$; then from the induction hypothesis after $k_1 + k_2$ steps the leading sub-matrix is lower triangular, and the only nonzero entries in the first row are $a_{1,1}^{k_1+k_2}$ and $a_{1,k+1}^{k_1+k_2}$. We return to Case 1 after $k_1 + k_2 + 1$ steps. In both cases the reduction to a lower triangular matrix is guaranteed in a finite number of preconditioning steps. ☑

From Lemmas 2.3 and 2.5 and Theorem 3.1 we get the following result.

Theorem 3.2. *Let A be an irreducible diagonally dominant Z-matrix with positive diagonal, and A_i the matrix defined as in Equation (4) with Gauss-Seidel regular splitting $A_i = M_i - N_i$. Then there exists a $k \in \mathbb{N}$ such that*

$$1 \; > \; \rho(M_0^{-1}N_0) \; > \; \rho(M_1^{-1}N_1) \; > \; \cdots > \; \rho(M_k^{-1}N_k) \; = \; 0.$$

We would like to address that if A is an H-matrix, then we can not claim that the descending property of the spectral radius stated in the theorem is true, but we still have for all j

$$\rho(M_j^{-1}N_j) < 1.$$

4. Numerical experiments

In this section we carry out several numerical experiments to illustrate the performance of the proposed preconditioner. We consider an implementation slightly different than the classical one which uses $\|x_n - x_{n-1}\|$ as stopping criterion. Here the l_2 norm of the current residual is used, that is $\|b - Ax_n\|_2$. To be more precise, computations are stopped when $\|b - Ax_n\|_2 \leq rtol\|b - Ax_0\|_2$, where $rtol$ is a user predefined relative tolerance.

4.1. Dense matrices

First we study the performance of the proposed preconditioner on dense diagonally dominant Z-matrices of various sizes which have been randomly generated. For comparison purposes, the matrix of size 10×10 is the one used in [15, p. 599]. In Table 1 we show the spectral radius for different matrix sizes and various number of preconditioning steps. 0 steps means that we consider the original matrix, that is, Gauss-Seidel iteration is performed on the original matrix. As predicted by the analysis of Theorem 3.2 the spectral radius decreases as the number of preconditioning steps increases.

	Preconditioning steps					
Matrix size	0	1	5	10	15	20
10	6.918e-01	6.230e-01	4.145e-01	1.531e-01	6.718e-02	1.664e-02
50	6.931e-01	6.805e-01	6.315e-01	5.741e-01	5.177e-01	4.673e-01
100	7.078e-01	7.015e-01	6.772e-01	6.471e-01	6.185e-01	5.902e-01
150	6.785e-01	6.739e-01	6.560e-01	6.345e-01	6.137e-01	5.934e-01
200	7.164e-01	7.131e-01	7.006e-01	6.856e-01	6.710e-01	6.565e-01

Table 1. Spectral radius of $\rho(\widehat{M}^{-1}\widehat{N})$.

In Table 2 we present the iteration count for the Gauss-Seidel method to reach a relative residual error of 10^{-10} in the l_2 norm. The right hand side was chosen such that the exact solution is the vector whose entries are all ones. The initial guess is the null vector. Observe that for each matrix as the number of preconditioning steps increases the number of iteration decreases, which is in agreement with the results presented in Table 1.

	Preconditioning steps					
Matrix size	0	1	5	10	15	20
10	63	50	27	13	9	6
50	63	61	51	42	36	31
100	67	66	60	54	49	44
150	60	59	55	51	48	45
200	70	69	65	62	58	55

Table 2. Iteration count for point version.

4.2. Block version of the $IpS_{max}(k)$ preconditioner

A natural generalization of the preconditioner can be obtained by simply partitioning the original matrix into blocks and applying a block version of the Gauss-Seidel iteration. This approach is in general more efficient than its classical point version. For example, the block version of Gauss-Seidel iteration will converge in only one iteration on a block triangular matrix. Numerical experiments presented recently in [3] show that block Gauss-Seidel outperforms the point version when it is used as a relaxation technique on a particular multilevel preconditioner designed for discontinuous Galerkin discretization. Some theoretical results of block relaxation techniques can be found in [1, 18]. The extension of Equation(1) to a block partitioned matrix will now read

$$S_{i,j} \;=\; \begin{cases} -A_{i,j}A_{j,j}^{-1}, & \text{if } j > i \text{ and } j = k_i, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

where

$$k_i \;=\; \min\left\{ j \;:\; \max_{i<k} \|A_{i,k}\| = \|A_{i,j}\| \right\},$$

and $S_{i,j}$ and $A_{i,j}$ are now blocks instead of scalars. The magnitude of a block B of size m is defined as $\|B\| = \max\{|b_{i,j}| : i,j = 1,\ldots,m\}$. The construction of matrix $\widehat{A} = (I + S_{max})A$ can be obtained again by simple row manipulations.

In Table 3 we show the iteration count for the block version of the preconditioner to reach a relative tolerance of 10^{-10} in the l_2 norm on a linear system using the diagonally dominant Z-matrix of size 100×100 of the previous experiment. The partitioning consists of blocks of the same size. When the block size is 1 the behavior of the block version should be the same as the point version, which is exactly what is shown in the first row of this table. The experiment shows the efficiency of the block version compared to its point version. Also the number of iterations decreases as the number of preconditioning steps increases.

Notice the rapid convergence of the block version for larger blocks. This can be explained as follows: since larger blocks are used, fewer preconditioning steps are required to annihilate matrix $N = -U$ of the Gauss-Seidel splitting. This is analogous to Theorem 3.2 for the point version.

	Preconditioning steps						
Block size	0	1	5	10	15	20	25
1	67	66	60	54	49	44	41
2	67	64	57	49	42	36	33
4	65	62	49	36	28	23	19
5	65	60	45	31	24	18	14
10	62	54	30	15	9	6	5
20	55	43	11	1	1	1	1
25	52	37	1	1	1	1	1
50	36	1	1	1	1	1	1

Table 3. Iteration count for block version.

4.3. Finite Volume matrices

In the next series of experiments the performance of our recursive preconditioner is numerically tested on matrices not satisfying the requirements of the theoretical analysis. We consider stiffness matrices obtained from the discretization of the following elliptic model problem:

$$-\nabla(K\nabla p) \;=\; 0, \quad \text{in } (0,\ 1)\times(0,\ 1),$$

where K is a symmetric positive definite tensor not necessarily continuous. The Finite Volume method is used as discretization method. The resulting matrices are weakly diagonally dominant Z-matrices.

Sand and shales problem

First we study the performance of the preconditioner on a classical benchmark porous media problem proposed by J.L. Durlofsky in [5]. The permeability field consists of only two materials, sand and shales; and is defined on a Cartesian grid consisting of 20×20 cells. In Figure 1 we illustrate the distribution of the permeability field; dark cells correspond to a permeability tensor $\mathcal{K} = 10^{-6} I_d$ and light cells to $\mathcal{K} = I_d$. Dirichlet boundary conditions are imposed at $x = 0$ where $p = 1$, and at $x = 1$, where $p = 0$. No flow boundary conditions are imposed at $y = 0$ and $y = 1$. This problem is difficult to solve since several interfaces are spread over the domain.

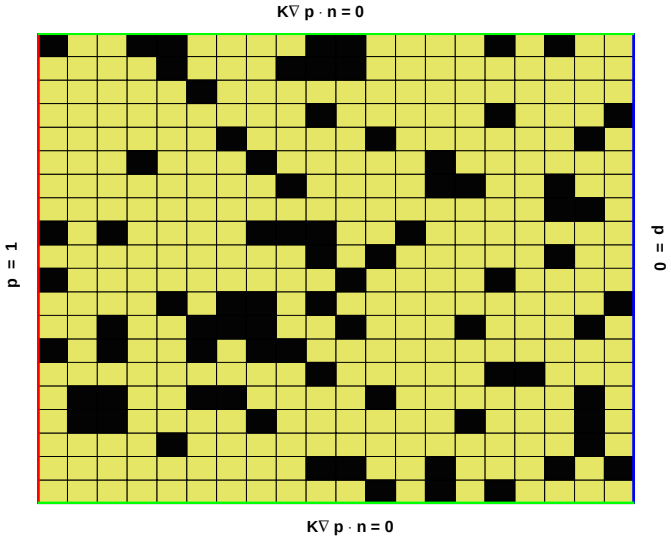

Figure 1. Description of Durlofsky's benchmark problem.

In Table 4 we present the iteration count for the preconditioned Gauss-Seidel method to reach a relative residual error of 10^{-6} in the l_2 norm on several globally refined meshes. The initial guess is the null vector and the maximum number of iteration has been set to 5000.

Since no convergence is observed in the third mesh within the prescribed maximum number of iterations, we illustrate in Table 5 the spectral radius $\rho(\widehat{M}^{-1}\widehat{N})$ for different

	Number of preconditioning steps						
cells	0	1	5	10	15	20	25
20×20	759	478	198	119	95	79	69
40×40	3338	2106	898	526	415	341	299
80×80	5000	5000	3521	2074	1646	1359	1208
160×160	5000	5000	5000	5000	5000	5000	4589

Table 4. Iteration for the Durlofsky's benchmark problem.

values of preconditioning steps, k. We observe a slowly decrease of the spectral radius as the number of preconditioning steps increases.

	Number of preconditioning steps						
cells	0	1	5	10	15	20	25
80×80	0.999358	0.998986	0.997546	0.995759	0.994585	0.993396	0.992520
160×160	0.999844	0.999755	0.999418	0.998992	0.998717	0.998441	0.998254

Table 5. Spectral radius of Durlofsky's benchmark problem.

Random isotropic permeability field

In the next numerical experiment we consider an isotropic random permeability field as shown in Figure 2. The permeability tensor of cell i is of the form $K_i = 10^{\alpha_i} I$, where α_i is a random number in the interval $[-6,\ 1]$. In this experiment the minimum and maximum values of α_i are -5.9674 and 0.9936 respectively.

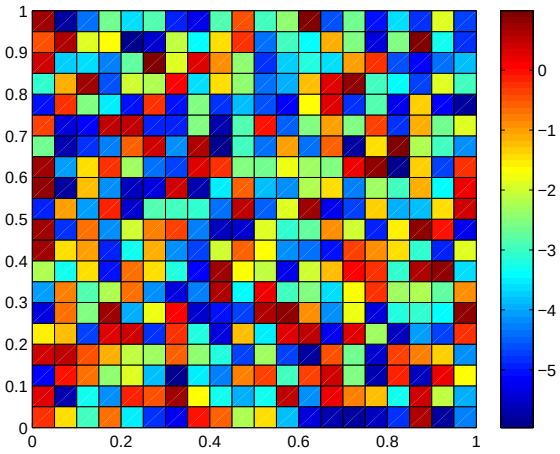

Figure 2. Random isotropic permeability distribution in logarithmic scale.

In Table 6 we present the iteration count for the preconditioned Gauss-Seidel method to reach a relative residual error of 10^{-6} in the l_2 norm. The initial guess is the null vector. Since this problem is more difficult to solve than the previous example, the maximum

number of iterations has been set to 5000. Observe that as we increase the number of preconditioning steps the number of iterations decreases. However, for meshes with a large number of cells, a larger number of preconditioned steps is still required since the stiffness matrix is more ill-conditioned for this type of problems.

	Number of preconditioning steps						
cells	0	1	5	10	15	20	25
400	5000	5000	1087	526	328	230	184
1600	5000	5000	5000	5000	5000	3932	2886
6400	5000	5000	5000	5000	5000	5000	5000

Table 6. Iteration count for the random isotropic permeability field.

Random anisotropic permeability field

We now increase the difficulty of our model problem by considering a random anisotropic permeability field. The permeability tensor is of the form

$$K_i = \begin{pmatrix} K_{11}^i & 0 \\ 0 & K_{22}^i \end{pmatrix},$$

where $K_{11}^i = 10^{\alpha_i}$ and $K_{22}^i = 10^{\beta_i}$ and the values of α_i and β_i are random numbers in the interval $[-6,\ 1]$.

In Table 7 we present the iteration count to reach a relative residual error of 10^{-6} in the l_2 norm. The initial guess is the null vector and the maximum number of iterations is 5000.

	Number of preconditioning steps						
cells	0	1	5	10	15	20	25
400	5000	2289	612	314	226	173	140
1600	5000	5000	5000	5000	3505	2599	1971
6400	5000	5000	5000	5000	5000	5000	5000

Table 7. Iteration count for the random anisotropic permeability field.

Since no convergence is observed in the third mesh within the prescribed maximum number of iterations, we illustrate in Figure 3 the behaviour of the norm of the residuals of the first 1000 iterations for different values, k. Again we observe a much better performance of the preconditioner as the number of preconditioning steps increases.

4.4. Computational cost

We end our numerical section with a brief discussion of the computational cost of the proposed preconditioner. When A is stored as a dense matrix the computational cost per iteration is reduced to a matrix vector multiplication. Hence the cost per iteration is proportional to n^2 where n is the matrix size. Consequently the most important factor in performance is only determined by the reduction of the spectral radius of the iteration matrix.

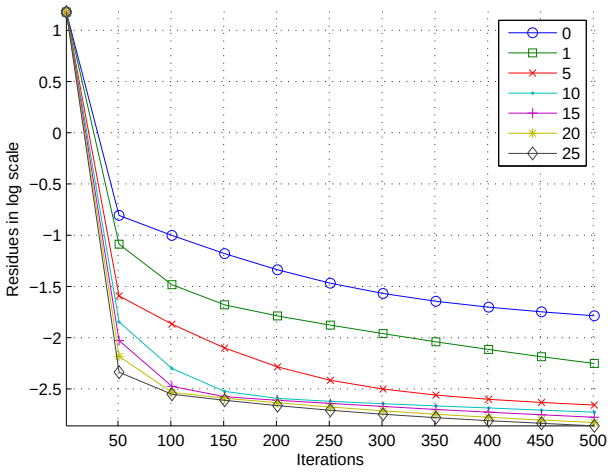

Figure 3. Residuals in in the l_2 norm, logarithmic scale.

For sparse matrices the analysis is not that straight. For example in Durlofsky's benchmark problem, the amount of non-zeros increases rapidly. Table 8 shows the ratio of the total number of non-zero entries between the matrix preconditioned with the point version of the IpS_{max} and the original matrix without preconditioning. For a small number of preconditioning steps the number of non-zero entries is relatively small; however, for a large number of steps the amount of storage becomes unacceptably expensive.

	Number of preconditioning steps					
cells	1	5	10	15	20	25
20×20	1.36	6.07	17.22	27.48	31.66	33.70
40×40	1.38	6.59	22.34	48.77	78.46	103.22
80×80	1.39	6.94	24.39	58.01	107.75	173.87
160×160	1.39	7.02	24.84	61.15	119.50	208.19

Table 8. Ratio of non-zero entries for the Durlofsky's benchmark problem.

5. Concluding remarks

In this work a preconditioner for accelerating the Gauss-Seidel relaxation method has been analyzed for diagonally dominant Z-matrices. The technique consists of performing a fixed number of $I + S_{max}$ preconditioning steps. Numerical experiments using a wide variety of matrices including matrices not covered by the theory show a good performance of this preconditioning technique. In addition a block version of the classical and the recursive $I + S_{max}$ preconditioner has been introduced for the first time. Although we have not presented any theoretical result proving the convergence of such technique, our numerical experiments show a superior performance compare to its point version.

We have shown that after a finite number of $I + S_{max}$ steps the Gauss-Seidel iteration matrix reduces to 0; however, finding an upper bound in terms of the matrix size remains

an open question and is subject of an ongoing research.

References

[1] Axelsson O., *Iterative Solution Methods,* Cambridge University Press, New York, 1994.

[2] Berman A. and Plemmons R.J., *Nonnegative Matrices in the Mathematical Sciences,* Classics in Applied Mathematics, 9, SIAM, 1994.

[3] Castillo P. and Sequeira F.A., "Computational aspects of the Local Discontinuous Galerkin method on unstructured grids in three dimensions", *Mathematical and Computer Modelling* 57 (2013), 2279–2288.

[4] Du J., Zheng B. and Wang L., "New iterative methods for solving linear systems", *J. Appl. Anal. Comput.* 1 (2011), no. 3, 351–360.

[5] Durlofsky L.J., "Accuracy of mixed and control volume finite element approximations to Darcy velocity and related quantities", *Water Resources Research* 30 (1994), no.4, 965–973.

[6] Gunawardena A.D., Jain S.K. and Snyder L., "Modified iterative methods for consistent linear systems", *Linear Algebra Appl.* 154-156 (1991), 123–143.

[7] Hadjidimos A., Noutsos D. and Tzoumas M., "More on modifications and improvements of classical iterative schemes for M-matrices", *Linear Algebra Appl.* 364 (2003), 253–279.

[8] Kohno T., Kotakemori H., Niki H. and Usui M., "Improving the modified Gauss-Seidel method for Z-matrices", *Linear Algebra Appl.* 267 (1997), 113–123.

[9] Kotakemori H., Harada K., Morimoto M. and Niki H., "A comparison theorem for the iterative method with preconditioner $(I + S_{max})$", *J. Comput. Appl. Math.* 145 (2002), no. 2, 373–378.

[10] Kotakemori H., Niki H. and Okamoto N., "Accelerated iterative method for Z-matrices", *J. Comput. Appl. Math.* 75 (1996), no. 1, 87–97.

[11] Liu Q., Chen G. and Cai J., "Convergence analysis of the preconditioned Gauss-Seidel method for H-matrices", *Comput. Math. Appl.* 56 (2008), no. 8, 2048–2053.

[12] Milaszewicz J.P., "Improving Jacobi and Gauss-Seidel iterations", *Linear Algebra Appl.* 93 (1987), 161–170.

[13] Mokari-Bolhassan M.E. and Trick T.N., "A new iterative algorithm for the solutions of large scale systems", *28th Midwest Symposium on Circuits and Systems Louisville, Kentucky,* 1985.

[14] Morimoto M., Harada K., Sakakihara M. and Sawami H., "The Gauss-Seidel iterative method with the preconditioning matrix $(I + S + S_M)$", *Japan J. Appl. Math.* 21 (2004), 25–34.

[15] Niki H., Harada K., Morimoto M. and Sakakihara M., "The survey of preconditioners used for accelerating the rate of convergence in the Gauss-Seidel method", *J. Comput. Applied. Math.* 164-165 (2004), 587–600.

[16] Shao J.L., Huang T.Z. and Zhang G.F., "Linear system based approach for solving some related problems of M-matrices", *Linear Algebra Appl.* 432 (2010), 327–337.

[17] Shen H., Shao X., Huang Z. and Li C., "Preconditioned Gauss-Seidel iterative method for Z matrices linear systems", *Bull. Korean. Math. Soc.* 48 (2011), 303–314.

[18] Varga R.S., *Matrix Iterative Analysis*, Springer Series in Computational Mathematics, Springer, Berlin, 2000.

[19] Zheng B. and Miao S.X., "Two new modified Gauss-Seidel methods for linear systems with M-matrices", *J. Comput. Appl. Math.* 233 (2009), 922–930.