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Spectral characterizations of sandglass graphs[☆]

Pengli Lu [a,*], Xiaogang Liu [b], Zhanting Yuan [a], Xuerong Yong [c]

[a] School of Computer and Communication, Lanzhou University of Technology, Lanzhou, 730050, Gansu, PR China

[b] Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, PR China

[c] Department of Mathematics, University of Puerto Rico at Mayaguez, P.O. Box 9018, PR 00681, USA

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ABSTRACT

The sandglass graph is obtained by appending a triangle to each pendant vertex of a path. It is proved that sandglass graphs are determined by their adjacency spectra as well as their *Laplacian* spectra.

1. Introduction

We consider undirected graphs with no loops or parallel edges. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $v_1, v_2, \ldots, v_n$ are indexed in the non-increasing order of degrees. Let $A(G)$ be the (0,1)-*adjacency matrix* of G and d_k be the degree of the vertex v_k. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G, where $D(G)$ is the $n \times n$ diagonal matrix with $\{d_1, d_2, \ldots, d_n\}$ as diagonal entries. Since $A(G)$ (resp. $L(G)$) is real and symmetric, its eigenvalues are real numbers and are called the *adjacency* (resp. *Laplacian*) *eigenvalues* of G. We denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (resp. $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$) the adjacency (resp. Laplacian) eigenvalues of G. The multiset of eigenvalues of $A(G)$ (resp. $L(G)$) is called the *adjacency* (resp. *Laplacian*) *spectrum* of G. Two graphs are said to be *cospectral* with respect to adjacency (resp. Laplacian) matrix if they have equal adjacency (resp. Laplacian) spectra. A graph is said to be determined by its spectrum if there is no other non-isomorphic graph with the same spectrum. Up until now, numerous examples of cospectral but non-isomorphic graphs are reported [1,2]. In Fig. 2, some bicyclic cospectral mates are shown, where Go_1 and Go_1' (see [3], pp. 27), Go_2 and Go_2' with trees attached on the black nodes, Go_3 and Go_3' with trees attached on the black nodes (see Section 3 for details) are respectively adjacency-cospectral mates, Go_4 and Go_4' (see [1]) are Laplacian-cospectral mates. But, only a few graphs with very special structures have been reported to be determined by their spectra [1,4–11].

In this paper, some spectral characterizations of the so-called sandglass graph will be discussed. The sandglass graph (shown in Fig. 1) is a graph, denoted by $G(C_3, C_3, P_r)$, obtained by appending a triangle to each pendant vertex of a path. Clearly, it is a bicyclic graph with $r + 4$ vertices and $r + 5$ edges. This paper is organized as follows: In Section 2, some available lemmas will be summarized. In Section 3, it will be proved that sandglass graphs are determined by their adjacency spectra. In Section 4, it will be proved that sandglass graphs are determined by their Laplacian spectra.

To fix notations, the disjoint union of two graphs G_1 and G_2 is noted $G_1 + G_2$.

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* Corresponding author.
E-mail addresses: lupengli88@gmail.com (P. Lu), liuxg@ust.hk (X. Liu), yuanzt@lut.cn (Z. Yuan), xryong@math.uprm.edu (X. Yong).

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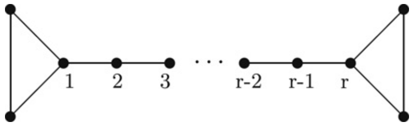

Fig. 1. Sandglass graph $G(C_3, C_3, P_r)$.

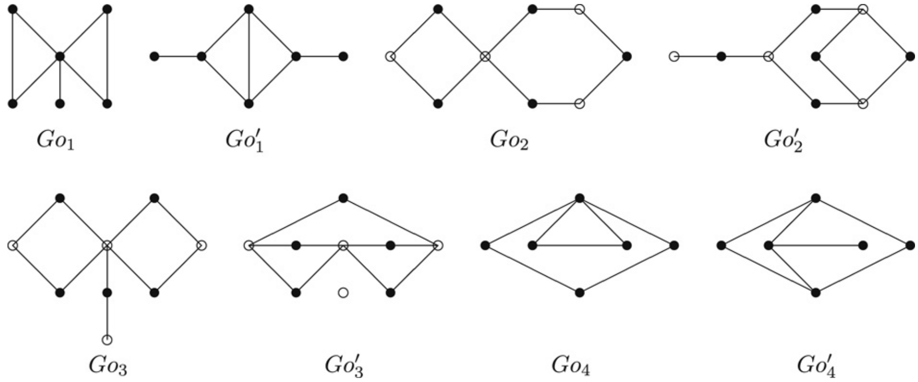

Fig. 2. Cospectral mates.

2. Preliminaries

Lemma 2.1 ([12]). *Let G be a graph on n vertices with adjacency characteristic polynomial $P_{A(G)}(\lambda) = \sum_{i=0}^{n} a_i \lambda^{n-i}$. Then*

$$a_i(G) = \sum_{S \in Li} (-1)^{k(S)} \cdot 2^{c(S)},$$

where Li denotes the set of Sachs graphs with i vertices (namely, the graph with its component being either the complete graph on two vertices denoted by K_2 or a cycle), $k(S)$ is the number of components of S and $c(S)$ is the number of cycles contained in S.

Lemma 2.2 ([1]). *Suppose that N is a symmetric $n \times n$ matrix with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Then the eigenvalues $\alpha'_1 \geq \alpha'_2 \geq \cdots \geq \alpha'_m$ of a principal submatrix of N of size m satisfy $\alpha_i \geq \alpha'_i \geq \alpha_{n-m+i}$ for $i = 1, 2, \ldots, m$.*

Lemma 2.3 ([13]). *In a simple graph, the number of closed walks of length 4 equals twice the number of edges plus four times of the number of paths on three vertices plus eight times of the number of 4-cycles.*

Lemma 2.4 ([1,14]). *Let G be a graph. For the adjacency matrix and the Laplacian matrix, the following can be deduced from the spectrum.*

(1) *The number of vertices.*
(2) *The number of edges.*
    For the adjacency matrix, the following follows from the spectrum.
(3) *The number of closed walks of any length.*
    For the Laplacian matrix, the following follows from the spectrum.
(4) *The number of components.*
(5) *The number of spanning trees.*
(6) *The sum of the squares of degrees of vertices.*

The following theorem relates the behavior of the spectral radius of a graph by subdividing an edge. An internal path of a graph G is an elementary path $x_0 x_1 \cdots x_k$ (i.e., $x_i \neq x_j$ for all $i \neq j$ but eventually $x_0 = x_k$) of G with $d(x_0) > 2$, $d(x_k) > 2$, $d(x_i) = 2$ for all other i's.

Lemma 2.5 ([7]). *Let G be a connected graph which is not isomorphic to W_n (shown in Fig. 3) and G_{uv} the graph obtained from G by subdividing the edge uv of G. If uv lies on an internal path of G, then $\lambda_1(G_{uv}) < \lambda_1(G)$.*

Lemma 2.6 ([15,16]). *Let G be a graph with $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$. Then*

$$d_1 + 1 \leq \mu_1 \leq \max\left\{ \frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j},\ v_i v_j \in E(G) \right\},$$

where m_i denotes the average of the degrees of the vertices adjacent to vertex v_i in G.

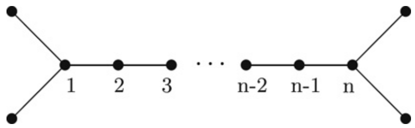

Fig. 3. The graph W_n.

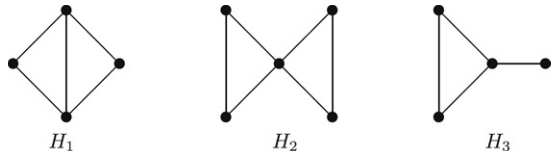

Fig. 4. The graphs H_1, H_2 and H_3.

Lemma 2.7. *Let G be a sandglass graph of $n = r + 4$ vertices shown in Fig. 1, where $r \geq 3$. Then*

(1) $\lambda_1(G) < 2.5$.
(2) *Let G' be a graph cospectral with G with respect to the adjacency matrix. Then graphs H_1 and H_2 (see Fig. 4) cannot be induced subgraphs of G'.*
(3) *For $r \geq 4$, $\lambda_2(G) > 2$.*
(4) $\lambda_3(G) < 2$.
(5) *The disjoint union of three cycles cannot be an induced subgraph of a graph cospectral with G.*

Proof. By using Matlab, we can see that $\lambda_1(H_1) = 2.5616$, $\lambda_1(H_2) = 2.5616$, $\lambda_1(H_3) = 2.1701$, $\lambda_1(G(C_3, C_3, P_3)) = 2.3429$ and $\lambda_1(G(C_3, C_3, P_4)) = 2.3028$.

(1) If $r = 3, r = 4$, $\lambda_1(G(C_3, C_3, P_r)) < 2.5$. If $r \geq 5$, we can obtain $G(C_3, C_3, P_r)$ from $G(C_3, C_3, P_{r-1})$ by subdividing any edge between the two vertices of degree 3. It follows from Lemma 2.5 that $\lambda_1(G(C_3, C_3, P_r)) < \lambda_1(G(C_3, C_3, P_{r-1})) < 2.5$. So $\lambda_1(G) < 2.5$.
(2) $\lambda_1(H_1) > 2.5$, $\lambda_1(H_2) > 2.5$. Let us suppose that graph H_1(resp. H_2) is an induced subgraph of G'. It follows from Lemma 2.2 that $\lambda_1(G') \geq \lambda_1(H_1)$ (resp. $\lambda_1(G') \geq \lambda_1(H_2)$). By (1) we have $\lambda_1(G') = \lambda_1(G) < 2.5$, a contradiction. So H_1 and H_2 cannot be induced subgraphs of G'.
(3) For $r = 4$, by using Matlab, $\lambda_2(G) = 2.1149 > 2$. For $r \geq 5$, let $M = M_1 + M_2$ be the subgraph obtained from G by deleting any vertex between the vertices of degree 3 but not adjacent to the vertices of degree 3, where M_1, M_2 are two components of M. Let H_3 be the graph shown in Fig. 4. Clearly, H_3 is an induced subgraph of M_1 and M_2. Since $\lambda_1(H_3) = 2.1701 > 2$, Lemma 2.2 implies that $\lambda_1(M_1) \geq \lambda_1(H_3) > 2$ and $\lambda_1(M_2) \geq \lambda_1(H_3) > 2$. Therefore $\lambda_2(M) > 2$. So $\lambda_1(G) \geq \lambda_1(M) \geq \lambda_2(G) \geq \lambda_2(M) > 2$, i.e., $\lambda_2(G) > 2$ for $r \geq 4$.
(4) Let G_{xy} be the subgraph by deleting two vertices of degree 3 from G. Then G_{xy} is the disjoint union of three paths. The adjacency eigenvalues of path P_n are $2\cos(\pi j/(n+1))$, $j = 1 \ldots n$, i.e., $\lambda_1(P_n) < 2$. So $\lambda_1(G_{xy}) = \lambda_1(P_n) < 2$. By Lemma 2.2, $\lambda_1(G) \geq \lambda_1(G_{xy}) \geq \lambda_3(G)$. Hence $\lambda_3(G) < 2$.
(5) It follows from (4). $\square$

Lemma 2.8. *Let G be a sandglass graph of n ($n \geq 10$) vertices with adjacency characteristic polynomial $P_{A(G)}(\lambda) = \sum_{i=0}^{n} a_i \lambda^{n-i}$. Then* $a_n(G) = \begin{cases} -3, n = 4k \\ 4, n = 4k+1 \\ 3, n = 4k-2 \\ -4, n = 4k-1 \end{cases}$ *, where $k \geq 3$ is an integer.*

Proof. Lemma 2.1 implies that $a_n(G) = \sum_{S \in L_n} (-1)^{k(S)} \cdot 2^{c(S)}$, where L_n denotes the set of Sachs graphs with n vertices.

Case 1. If n is even, then $L_n = \{S_1, S_2\}$, where $S_1 = 2C_3 + \frac{n-6}{2} K_2$ and $S_2 = \frac{n}{2} K_2$. Since $k(S_1) = 2 + \frac{n-6}{2} = \frac{n-2}{2}$, $c(S_1) = 2$, $k(S_2) = \frac{n}{2}$ and $c(S_2) = 0$, then $a_n(G) = (-1)^{k(S_1)} \cdot 2^{c(S_1)} + (-1)^{k(S_2)} \cdot 2^{c(S_2)} = 4(-1)^{\frac{n-2}{2}} + (-1)^{\frac{n}{2}} = \begin{cases} -3, n = 4k \\ 3, n = 4k-2 \end{cases}$, where $k \geq 3$ is an integer.

Case 2. If n is odd, then $L_n = \{S_1, S_2\}$, where S_1 and S_2 are isomorphic to $C_3 + \frac{n-3}{2} K_2$. Since $k(S_1) = k(S_2) = 1 + \frac{n-3}{2} = \frac{n-1}{2}$, $c(S_1) = c(S_2) = 1$, then $a_n(G) = (-1)^{k(S_1)} \cdot 2^{c(S_1)} + (-1)^{k(S_2)} \cdot 2^{c(S_2)} = 4(-1)^{\frac{n-1}{2}} = \begin{cases} 4, n = 4k+1 \\ -4, n = 4k-1 \end{cases}$, where $k \geq 3$ is an integer. $\square$

Lemma 2.9. *Let W be a graph of n ($n \geq 10$) vertices shown in Fig. 6 with adjacency characteristic polynomial $P_{A(W)}(\lambda) = \sum_{i=0}^{n} a_i \lambda^{n-i}$. Then* $a_n(W) = \begin{cases} -16, n = 4k \\ -6, n = 4k+1 \\ 0, n = 4k-2 \\ -10, n = 4k-1 \end{cases}$ *, where $k \geq 3$ is an integer.*

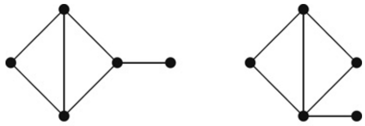

Fig. 5. The graphs G_1 and G_2.

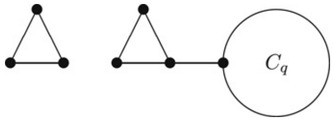

Fig. 6. The graph W.

Proof. Lemma 2.1 implies that $a_n(W) = \sum_{S\in L_n}(-1)^{k(S)} \cdot 2^{c(S)}$, where L_n denotes the set of Sachs graphs with n vertices.

Case 1. If n is even, clearly, the length of cycle C_q is also even. Then $L_n = \{S_1, S_2, S_3\}$, where $S_1 = 2C_3 + C_q$, S_2 and S_3 are isomorphic to $2C_3 + \frac{n-6}{2}K_2$. Since $k(S_1) = 3$, $c(S_1) = 3$, $k(S_2) = k(S_3) = 2 + \frac{n-6}{2}$ and $c(S_2) = c(S_3) = 2$, then $a_n(W) = \sum_{i=1}^{3}(-1)^{k(S_i)} \cdot 2^{c(S_i)} = -8 + 8(-1)^{\frac{n-2}{2}} = \begin{cases}-16, n = 4k\\ 0, n = 4k-2\end{cases}$, where $k \geq 3$ is an integer.

Case 2. If n is odd, clearly, the length of cycle C_q is also odd. Then $L_n = \{S_1, S_2\}$, where $S_1 = 2C_3 + C_q$ and $S_2 = C_3 + \frac{n-3}{2}K_2$. Since $k(S_1) = 3$, $c(S_1) = 3$, $k(S_2) = 1 + \frac{n-3}{2} = \frac{n-1}{2}$ and $c(S_2) = 1$. Then $a_n(W) = \sum_{i=1}^{2}(-1)^{k(S_i)} \cdot 2^{c(S_i)} = -8 + 2(-1)^{\frac{n-1}{2}} = \begin{cases}-6, n = 4k+1\\ -10, n = 4k-1\end{cases}$, where $k \geq 3$ is an integer. □

3. Sandglass graphs are determined by their adjacency spectra

First, we will prove that the graphs $Go2$ and $Go2'$, $Go3$ and $Go3'$ in Fig. 2 and their complements are cospectral with respect to the adjacency matrix, respectively.

Theorem 3.1. *The graphs $Go2$ and $Go2'$, $Go3$ and $Go3'$ given in Fig. 2 are cospectral with respect to the adjacency matrix, respectively. And the same is true for their complements.*

Proof. Consider the white vertices in $Go2$ and $Go2'$ (resp. $Go3$ and $Go3'$) in Fig. 2. For each white vertex v, delete the edges between v and the black neighbors, and insert edges between v and the other black vertices. It is easily checked that this operation transforms $Go2$ into $Go2'$ (resp. $Go3$ into $Go3'$). Godsil and McKay (see [17], this operation is called Godsil–McKay switching) have shown that this operation leaves the adjacency spectrum of the graph and its complements unchanged. □

Theorem 3.2. *A sandglass graph is determined by its adjacency spectrum.*

Proof. Let G be a sandglass graph with $n = r + 4$ vertices shown in Fig. 1. Suppose G' is cospectral with G with respect to the adjacency matrix. By (1) and (2) of Lemma 2.4, G' has $r + 4$ vertices and $r + 5$ edges. By (3) of Lemma 2.4, G and G' both have exactly two triangles.

First, we consider the special case of G: $r = 1$.

If $r = 1$, G' is a graph with 5 vertices, 6 edges and two triangles. Hence the two triangles belong to one connected component (otherwise the number of vertices of G' is more than 5). So the possible graph of G' is isomorphic to G_1, G_2 (shown in Fig. 5) or G.

Clearly, the adjacency characteristic polynomial of G_1(or G_2) is not equal the adjacency characteristic polynomial of G. Therefore G' is isomorphic to G.

Second, we consider the general case: $r \geq 2$.

If $r \geq 2$, suppose that G' has n_i vertices of degree i, for $i = 0, 1, 2, \ldots, \Delta'$, where Δ' is the largest degree of G'. Properties (1) and (2) of Lemma 2.4 imply that

$$n_0 + \sum_{i=1}^{\Delta'} n_i = r + 4, \tag{3.1}$$

$$\sum_{i=1}^{\Delta'} i n_i = 2(r + 5). \tag{3.2}$$

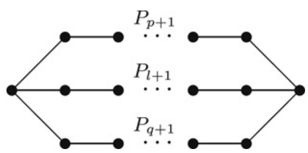

Fig. 7. The bicyclic graph BH_1 with $p + l + q - 1$ vertices.

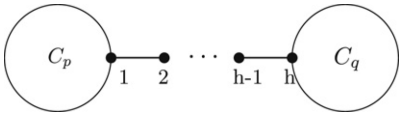

Fig. 8. The bicyclic graph BH_2.

By (3.1) and (3.2),

$$\sum_{i=1}^{\Delta'}(i-1)n_i = r + 6 + n_0. \tag{3.3}$$

By (3) of Lemma 2.4, G and G' have the same number of close walks of length 4. Let N_{C_4} be the number of 4-cycles in G', Lemma 2.3 implies that

$$\sum_{i=1}^{\Delta'}\binom{i}{2}n_i + 2N_{C_4} = 2\binom{3}{2} + (r+2)\binom{2}{2}. \tag{3.4}$$

By (3.3) and (3.4),

$$2n_0 + \sum_{i=1}^{\Delta'}(i^2 - 3i + 2)n_i + 4N_{C_4} = 4,$$

i.e.,

$$2n_0 + 2n_3 + 6n_4 + \sum_{i=5}^{\Delta'}(i^2 - 3i + 2)n_i + 4N_{C_4} = 4.$$

Then $n_i = 0$ for $i = 4, \ldots, \Delta'$, i.e., the largest degree of G' is at most 3. If $N_{C_4} = 1$, we obtain that $n_3 = 0$, $n_0 = 0$. By (3.1) and (3.2), $n_2 = r + 6$, $n_1 = -2$, a contradiction. Therefore $N_{C_4} = 0$. Consider the following cases.

(1) $n_3 = 0$, $n_0 = 2$. By (3.1) and (3.2), $n_2 = r + 8$, $n_1 = -6$, a contradiction.
(2) $n_3 = 1$, $n_0 = 1$. By (3.1) and (3.2), $n_2 = r + 5$, $n_1 = -3$, a contradiction.
(3) $n_3 = 2$, $n_0 = 0$. By (3.1) and (3.2), $n_2 = r + 2$, $n_1 = 0$. Therefore G' is a graph of $r + 4$ vertices and $r + 5$ edges with exactly two vertices of degree 3 and $r + 2$ vertices of degree 2 and containing exactly two triangles. If $r = 2$, obviously, G' is isomorphic to G. Hence, we only need to consider $r \geq 3$.

Suppose G' is not connected, then graph G' consists of a bicyclic graph BH_1 (shown in Fig. 7) or BH_2 (shown in Fig. 8) and at most one other cycle (Lemma 2.7 (4)). Consider the following cases.

Case 1. One and only one of the two triangles is in the bicyclic component. Then $G' = BH_1 + C_3$ for $l = 1$, $p = 2$ and $q \geq 3$ or $G' = BH_2 + C_3$ for $p = 3$ and $q \geq 4$.

Case 1.1 $G' = BH_1 + C_3$ for $l = 1$, $p = 2$ and $q \geq 3$. Let M be the subgraph obtained by deleting one of the vertices of degree 3 from BH_1, by Lemma 2.2, $\lambda_1(BH_1) \geq \lambda_1(M) \geq \lambda_2(BH_1)$. Since M is a path and its largest adjacency eigenvalue is strictly less than 2, then $2 > \lambda_2(BH_1)$. Then for $|V(G')| \geq 8$, $\lambda_1(G') = \lambda_1(BH_1) > 2$ and $\lambda_2(G') = \lambda_1(C_3) = 2$, a contradiction to (3) of Lemma 2.7.

Case 1.2 $G' = BH_2 + C_3$ for $p = 3$ and $q \geq 4$.

(1) $h = 1$. The largest degree of G' is 4, a contradiction.
(2) $h = 2$. Clearly, G' is the graph W shown in Fig. 6. Since G and W have the same adjacency characteristic polynomial, then $a_i(G) = a_i(W)$ for $i = 0, 1, \ldots, n$. But Lemmas 2.8 and 2.9 imply that $a_n(G) \neq a_n(W)$, a contradiction.
(3) $h \geq 3$. It is impossible by (5) of Lemma 2.7.

Case 2. Both of the two triangles are in the bicyclic component. Since by (2) of Lemma 2.7, H_1 and H_2 cannot be induced subgraphs of G', then $G' \neq H_1 + C_{m_1}$ and $G' \neq H_2 + C_{m_2}$, where $m_1 = r$ and $m_2 = r - 1$. Then $G' = BH_2 + C_{m_3}$ for $p = 3$, $q = 3$, $h \geq 2$ and $m_3 \geq 4$.

Case 2.1 $h = 2$. Clearly, the two vertices of degree 3 in BH_2 are adjacent. By using Matlab, we can get $\lambda_1(BH_2) = 2.4142$, $\lambda_2(BH_2) = 1.7321$. So $\lambda_2(G') = \lambda_1(C_{m_3}) = 2$, a contradiction to (3) of Lemma 2.7.

Case 2.2 $h \geq 3$. It is impossible for (5) of Lemma 2.7. Hence G' is connected. Therefore G' is isomorphic to G. $\square$

4. Sandglass graphs are determined by their Laplacian spectra

Theorem 4.1. *A sandglass graph is determined by its Laplacian spectrum.*

Proof. Let G be a sandglass graph with $n = r+4$ vertices shown in Fig. 1. If $r = 1$, G is a multi-fan graph $(P_2+P_2) \times b$, which is DS by its Laplacian spectrum (see [6]). We consider $r \geq 2$. Suppose G' is cospectral with G with respect to the Laplacian matrix. By (1), (2) and (4) of Lemma 2.4, G' is a connected graph with $r+4$ vertices and $r+5$ edges.

By using Matlab, we can see that $\mu_1(G(C_3, C_3, P_2)) = 4.5616$, $\mu_1(G(C_3, C_3, P_3)) = 4.4142$. For $r > 3$, Lemma 2.6 implies that $4 \leq \mu_1(G) \leq 4.8$. So the largest degree of graph G' is at most 3. Suppose that G' has n_i vertices of degree i, for $i = 1, \dots, \Delta'$, where $\Delta' \leq 3$ is the largest degree of G'. (1), (2) and (6) of Lemma 2.4 imply the following equations:

$$\sum_{i=1}^{\Delta'} n_i = r+4, \tag{4.1}$$

$$\sum_{i=1}^{\Delta'} i n_i = 2(r+5), \tag{4.2}$$

$$\sum_{i=1}^{\Delta'} i^2 n_i = 4(r+2) + 9 \times 2. \tag{4.3}$$

If $\Delta' \leq 2$, by (4.1) and (4.2), $n_2 = r+6$, $n_1 = -2$, a contradiction. So $\Delta' = 3$. By (4.1)–(4.3), $n_1 = 0$, $n_2 = r+2$, $n_3 = 2$. Therefore G' is BH_1 for $l \geq 1$, $p \geq q \geq 2$ or BH_2 for $p \geq q \geq 3$.

If $G' = BH_1$, (1) and (5) of Lemma 2.4 imply that $p+l+q-1 = r+4$, $lp+lq+pq = 9$. Clearly, $l \in \emptyset, p \in \emptyset, q \in \emptyset$, a contradiction.

If $G' = BH_2$, the number of spanning trees of G' is $pq = 9$. Hence $p = 3$, $q = 3$, *i.e.*, G' is isomorphic to G. $\square$

For a graph, its Laplacian eigenvalues determine the eigenvalues of its complement [15], so the complement of the sandglass graphs are determined by their Laplacian spectra.

5. Conclusion

In this paper, sandglass graphs are proved to be determined by their adjacency spectra as well as their Laplacian spectra. Which bicyclic graphs are determined by their spectra? And in particular, are the graphs obtained by joining two cycles by a path determined by their spectra? The answer is unknown. Therefore, we should try to find some new methods to solve these problems.

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