The Number of Spanning Trees in a Class of Double Fixed-Step Loop Networks

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In this article, we develop a method to count the number of spanning trees in certain classes of double fixed-step loop networks with nonconstant steps. More specifically our technique finds the number of spanning trees in $\vec{C}_n^{p,q}$, the double fixed-step loop network with n vertices and jumps of size p and q, when $n = d_1 m$, and $q = d_2 m + p$ where d_1, d_2, and p are arbitrary parameters and m is a variable. © 2008 Wiley Periodicals, Inc. NETWORKS, Vol. 52(2), 69–77 2008

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1. INTRODUCTION

A directed circulant graph $\vec{C}_n^{s_1,s_2,\dots,s_k}$ is a digraph on n vertices $0, 1, 2, \dots, n-1$; for each vertex $i(0 \leq i \leq n-1)$, there are k arcs from i to vertices $i + s_1, i + s_2, \dots, i + s_k \pmod{n}$. (Figure 1 illustrates $\vec{C}_6^{4,5}$.) A double fixed-step loop network is a directed circulant graph $\vec{C}_n^{p,q}$ in which each vertex has exactly two arcs leaving it. This kind of network appears in the design and analysis of local area networks, multimodule memory organizations, and parallel processing architectures [1, 6, 9]. Parameters of these graphs such as diameter and average distance, which are closely related to the network

bandwidth, have been considered recently. For example, the case $\vec{C}_n^{1,n-1}$ and the case $\vec{C}_n^{1,n-2}$, the so called daisy-chain loop, were investigated by Liu [12]. Some generalizations to infinite classes of double fixed-step loop networks with minimum diameter were explored by Erdös and Hsu [5]. More recent results can be found in [1, 6, 9] and their references.

In this article we address the question of counting the number of spanning trees in such digraphs. A spanning tree in a digraph is a rooted tree with directed paths from the root to all vertices. The number of spanning trees in a digraph or graph is an important, well studied quantity [4]. This parameter characterizes the reliability of networks. There is a classic result known as the Matrix Tree Theorem [11], which expresses the number of spanning trees $T(G)$ of a graph G as a function of the determinant of a matrix that can be easily constructed from G's incidence matrix. However, in practice, counting the spanning trees by calculating the determinant is infeasible for large graphs. For this reason, researchers have paid much attention to developing techniques or deriving formulas for analyzing the number of spanning trees. For some special classes of graphs, it is possible to give explicit, simple formulae for the number of trees. For example, if G is the complete graph K_n, then *Cayley's tree formula* [8] states that $T(K_n) = n^{n-2}$. Other special cases can be found in [3, 14, 17].

The asymptotic behavior of $T(\vec{C}_n^{1,q})$ has been derived in [15]. A closed formula for $T(\vec{C}_n^{1,2})$ was proved in [13] where it was also proved that $T(\vec{C}_n^{1,2}) \geq T(\vec{C}_n^{k,l})$ for any different positive integers k, l.

For fixed integers $s_1, s_2, \dots, s_k$, $1 \leq s_1 < s_2 < \cdots < s_k$, it was proved in [14] and [17] that $T(\vec{C}_n^{s_1,s_2,\dots,s_k}) = na_n$, where a_n satisfies a linear recurrence relation of order 2^{s_k-1}. This recurrence relation can be exactly derived by using the Matrix

Tree Theorem to calculate $a_n = T(\vec{C}_n^{s_1,s_2,\ldots,s_k})/n$ for $n = 1, 2, 3, \ldots, 2^{s_k}$ which gives the initial conditions and enough information to solve for the coefficients of the recurrence relation.

The technique is not applicable, though, when the jumps s_i *vary* with n. To the best of our knowledge, only a few very special cases of such graphs, e.g., the Möbius ladder [3], have been studied. Recently, Golin et al. [7] proved that when the jumps are linear in the graph size, the number of spanning trees (as a function of the graph size) also satisfies a linear recurrence relation. Their proof was an existence one, though. Constructing the recurrence relation based on their existence proof would require calculating a very large number of initial values and is thus infeasible except for a few simple cases. In this article, we will consider the number of spanning trees in a class of double fixed-step loop networks with jumps linear in the graph size. More specifically, we will develop a method for calculating $T(\vec{C}_n^{p,q})$ when $n = d_1 m$, and $q = d_2 m + p$ where d_1, d_2, and p are arbitrary parameters and m is a variable. In the next section we introduce our technique by developing all the necessary mathematical tools; in Section 3 we illustrate the technique by deriving the following series of formulas:

$$T(\vec{C}_{2m}^{1,m+1}) = m2^{2m-1},$$

$$T(\vec{C}_{2m}^{2,m+2}) = \begin{cases} m2^{2m-1} & \text{if } 2 \nmid m, \\ 0 & \text{otherwise,} \end{cases}$$

$$T(\vec{C}_{4m}^{1,m+1}) = m\left(2^{4m-1} + 2^{3m-1} + 2^{\frac{7m}{2}} \cos\frac{m-2}{4}\pi\right),$$

$$T(\vec{C}_{3m}^{1,m+1}) = m\left(2^{3m-1} + 2^{m-1} - 2^{2m} \cos\frac{m-2}{3}\pi\right),$$

$$T(\vec{C}_{3m}^{1,2m+1}) = m\left(2^{3m-1} + 2^{m-1} + 2^{2m} \cos\frac{m-1}{3}\pi\right),$$

$$T(\vec{C}_{3m}^{2,m+2}) = \begin{cases} m\left(2^{3m-1} + 2^{m-1} + 2^{2m} \cos\frac{m-1}{3}\pi\right) & \text{if } 2 \nmid m, \\ 0 & \text{otherwise.} \end{cases}$$

$$T(\vec{C}_{3m}^{2,2m+2}) = \begin{cases} m\left(2^{3m-1} + 2^{m-1} - 2^{2m} \cos\frac{m-2}{3}\pi\right) & \text{if } 2 \nmid m, \\ 0 & \text{otherwise.} \end{cases}$$

$$T(\vec{C}_{3m}^{3,m+3}) = T(\vec{C}_{3m}^{3,2m+3}) = \begin{cases} m\left(2^{3m-1} + 2^{m-1} - 2^{2m} \cos\frac{m}{3}\pi\right) & \text{if } 3 \nmid m, \\ 0 & \text{otherwise.} \end{cases}$$

We point out that the basis of our method is not the fact that our digraphs are circulants but two consequences of the fact that they are circulants: (i) they are regular digraphs and (ii) the jth eigenvalue of the digraph can be expressed as $f(\varepsilon^j), j = 1, 2, \ldots n$, where $f(x)$ is a polynomial and $\varepsilon = e^{\frac{2\pi i}{n}}$ [2]. Our technique could be extended to other graph classes with these properties as well. We also point out that our technique is restricted to counting the number of labeled spanning trees and cannot count the number of nonisomorphic spanning trees. This is because the Matrix Tree Theorem counts the number of labeled spanning trees and our technique is an application of the Matrix Tree Theorem. An interesting open question would be to develop a method for counting the number of nonisomorphic spanning trees for these graph classes.

2. THE BASIC IDEA

Our approach is based on the following result:

Lemma 1 (Zhang and Yong [16]). *For any positive integers n, p and q*

$$T(\vec{C}_n^{p,q}) = \prod_{j=1}^{n-1}(2 - \varepsilon^{pj} - \varepsilon^{qj}), \tag{1}$$
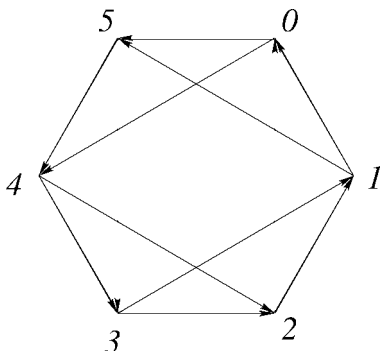
where $\varepsilon = e^{\frac{2\pi i}{n}}$.

FIG. 1.   The directed circulant graph $\vec{C}_6^{4,5}$.

This lemma and similar ones are at the basis of most analyses of the number of spanning trees of circulant graphs. They come from combining the Matrix Tree Theorem with observations concerning the eigenvalues of the adjacency matrices of circulant graphs; see e.g., [17] for more details. While the lemma does provide a formula for $T(\vec{C}_n^{p,q})$ it is not a particularly useful one. The rest of this paper will be devoted to transforming (1) into something more interesting.

Lemma 2. *Let n, p and q be any positive integers and* $\varepsilon = e^{\frac{2\pi i}{n}}$. *Define* $\delta_1, \delta_2, \ldots, \delta_n$ *so that*

$$f(x) = \prod_{j=1}^{n}(x - \varepsilon^{pj} - \varepsilon^{jq})$$

$$= x^n - \delta_1 x^{n-1} + \delta_2 x^{n-2} - \cdots + (-1)^n \delta_n.$$

Then

$$T(\vec{C}_n^{p,q}) = f'(2) = n2^{n-1} + \sum_{j=1}^{n-1}(-1)^j \delta_j (n-j)2^{n-j-1}. \quad (2)$$

Proof. Note that $f(2) = 0$, and from Lemma 1 we know

$$T(\vec{C}_n^{p,q}) = \prod_{j=1}^{n-1}(2 - \varepsilon^{pj} - \varepsilon^{qj})$$

$$= \lim_{x \to 2}\prod_{j=1}^{n-1}(x - \varepsilon^{pj} - \varepsilon^{qj})$$

$$= \lim_{x \to 2}\frac{f(x)}{x-2} = \lim_{x \to 2}\frac{f(x) - f(2)}{x-2} = f'(2).$$

∎

To find $T(\vec{C}_n^{p,q})$ our approach will therefore be to find $\delta_1, \delta_2, \ldots, \delta_{n-1}$ and substitute them into (2). Our main tool in calculating the δ_i will be Newton's formulae [10] which states the following:

- *Let* $x_1, x_2, \ldots, x_n$ *be arbitrary values.*
- *Let* $\delta_1, \delta_2, \ldots, \delta_n$ *be the coefficients of*

$$f(x) = (x - x_1)(x - x_2)\cdots(x - x_n) = x^n - \delta_1 x^{n-1}$$
$$+ \cdots + (-1)^n \delta_n.$$

- *For* $i = 1, 2, \ldots, n$, *define* $\mathcal{S}_i = x_1^i + x_2^i + \cdots + x_n^i$.
- *Then Newton's formulae are, for* $i = 1, 2, \ldots, n$:

$$\mathcal{S}_i - \delta_1\mathcal{S}_{i-1} + \delta_2\mathcal{S}_{i-2} + \cdots + (-1)^{i-1}\delta_{i-1}\mathcal{S}_1 + (-1)^i i\delta_i = 0. \quad (3)$$

More specifically, Newton's formulae permit us to derive the δ_i through knowledge of the $\mathcal{S}_i$. Note that the roots of $f(x)$ in Lemma 2 are $(\varepsilon^{kp} + \varepsilon^{kq}), k = 1, 2, \ldots, n$. In the following lemma and the sequel we use C_n^j to denote $\binom{n}{j}$.

Lemma 3. *Let*

$$S_i = \sum_{k=1}^{n}(\varepsilon^{kp} + \varepsilon^{kq})^i, i = 1, 2, \ldots, n,$$

where $\varepsilon = e^{\frac{2\pi i}{n}}$. *Then, for* $1 \le i \le n$:

$$S_i = n \sum_{\substack{0 \le j \le i \\ pi+(q-p)j=0(\bmod\, n)}} C_i^j.$$

Proof. For $1 \le i \le n$

$$S_i = \sum_{k=1}^{n}(\varepsilon^{kp} + \varepsilon^{kq})^i$$

$$= \sum_{k=1}^{n}\sum_{j=0}^{i}C_i^j \varepsilon^{kpi+k(q-p)j}$$

$$= \sum_{j=0}^{i}C_i^j\left(1 + \varepsilon^{pi+(q-p)j} + \varepsilon^{2(pi+(q-p)j)} + \cdots \right.$$

$$\left. + \varepsilon^{(n-1)(pi+(q-p)j)}\right)$$

$$= n \sum_{\substack{0 \le j \le i \\ pi+(q-p)j=0(\bmod\, n)}} C_i^j.$$

∎

In the most general case of arbitrary n, p, q this lemma does not help us much since the sums involved are quite complicated. However, in the particular cases in which $q = d_2 m + p$, $n = d_1 m$ where p and $d_2 < d_1$ are fixed and m grows, we can greatly simplify this sum, as shown in the next corollary:

Corollary 1. *Let* p, d_1 *and* d_2 *with* $d_2 < d_1$ *be fixed,* $n = d_1 m$, *and* $q = d_2 m + p$. *Set*

$$S_i = \sum_{k=1}^{n}(\varepsilon^{kp} + \varepsilon^{kq})^i, i = 1, 2, \ldots, n,$$

where $\varepsilon = e^{\frac{2\pi i}{n}}$. *Further define*

$$\begin{aligned}
\alpha &= \gcd(p, m), & p' &= p/\alpha, \\
\beta &= \gcd(d_1, d_2), & & \\
\gamma &= \gcd(\beta, p'), & p'' &= p'/\gamma, \\
d_1' &= d_1/\beta, & & \\
d_2' &= d_2/\beta & &
\end{aligned}$$

and let $\bar{d_2'}$ *be such that* $d_2'\bar{d_2'} \equiv 1 \pmod{d_1'}$. *Then*

$$S_i = \begin{cases} 0 & \text{if } \frac{\beta}{\gamma\alpha}m \nmid i, \\ n \sum_{t=0}^{\lfloor(i-x)/d_1'\rfloor} C_i^{x+td_1'} & \text{if } i = \ell\frac{\beta}{\gamma\alpha}m, \ell = 1, 2, \ldots, \frac{d_1\alpha\gamma}{\beta}, \end{cases} \quad (4)$$

where $x = (-\ell p''\bar{d_2'}) \bmod d_1'$.

Proof. Recall from Lemma 3 that $S_i = n \sum_{\substack{0 \le j \le i \\ pi+(q-p)j \equiv 0\,(\bmod\ n)}} C_i^j$. In what follows we examine, for fixed i, which j satisfy the condition

$$pi + (q - p)j \equiv 0 \quad (\bmod\ n). \qquad (5)$$

Relation (5) is equivalent to

$$pi + d_2mj \equiv 0 \quad (\bmod\ d_1m)$$

which can only be satisfied if $m|pi$ or, since $\alpha = gcd(m, p)$, if $\frac{m}{\alpha}|i$. We may therefore assume that $i = \ell\frac{m}{\alpha}, \ell = 1, 2, \ldots, \alpha d_1$. Then

$$\text{(5) is satisfied} \iff \ell\frac{m}{\alpha}p + d_2mj \equiv 0 \quad (\bmod\ md_1)$$
$$\iff \ell p' + d_2j \equiv 0 \quad (\bmod\ d_1)$$
$$\iff d_2j \equiv (-\ell p') \quad (\bmod\ d_1).$$

Since $\beta = gcd(d_1, d_2)$, if $\beta \nmid \ell p'$ this last condition cannot be satisfied so, if $S_i \neq 0$, then $\beta|\ell p'$ or $\frac{\beta}{\gamma}|\ell\frac{p'}{\gamma}$. Since $\gamma = gcd(\beta, p')$, we may assume that $\ell = \frac{\beta}{\gamma}\ell'$ for some integer ℓ'. This in turn implies that

$$i = \ell\frac{m}{\alpha} = \ell'\frac{\beta}{\gamma}\frac{m}{\alpha}$$

and

$$\text{(5) is satisfied} \iff d_2j \equiv (-\ell p') \quad (\bmod\ d_1)$$
$$\iff d_2j \equiv \left(-\ell'\beta\frac{p'}{\gamma}\right) \quad (\bmod\ d_1)$$
$$\iff d_2'j \equiv (-\ell'p'') \quad (\bmod\ d_1')$$
$$\iff j \equiv (-\ell'p'')\bar{d}_2' \quad (\bmod\ d_1')$$

from which (4) follows. ∎

We now note that even though we proved the corollary in full generality, in our spanning tree application we will not need this full generality. More specifically, we have the following result:

Lemma 4. *If $gcd(p, q, n) > 1$ then*

$$T(\vec{C}_n^{p,q}) = 0. \qquad (6)$$

In particular, given p, d_1, d_2, let $\alpha = gcd(p, m)$ and $\delta = gcd(d_1, d_2, p)$. Then, if either $\alpha > 1$ or $\delta > 1$,

$$T(\vec{C}_{d_1m}^{p,d_2m+p}) = 0.$$

Proof. To prove (6) note that if $gcd(n, p, q) > 1$ and (u, v) is an arc in $\vec{C}_n^{p,q}$ then $u \equiv v \pmod{gcd(n, p, q)}$. This implies that if u', v' are any two vertices in $\vec{C}_n^{p,q}$ and there is a path from u' to v' then $u' \equiv v' \pmod{gcd(n, p, q)}$. This in

turn implies that there is no one vertex in $\vec{C}_n^{p,q}$ from which it is possible to reach all of the vertices so $\vec{C}_n^{p,q}$ does not contain any spanning tree.

(An alternative proof would be to note that, setting $j = \frac{n}{gcd(n,p,q)}$ would give $\varepsilon^{jp} = \varepsilon^{jq} = 1$ so $(2 - \varepsilon^{jp} - \varepsilon^{jq}) = 0$ and, from (1), $T(\vec{C}_n^{p,q}) = \prod_{j=1}^{n-1}(2 - \varepsilon^{pj} - \varepsilon^{qj}) = 0$.)

To prove the second part of the lemma simply note that if either $\alpha > 1$ or $\delta > 1$ then $gcd(p, d_2m + p, d_1m) > 1$. ∎

In the sequel we may therefore assume that (i) $\alpha = gcd(p, m) = 1$ so $p' = p/\alpha = p$ and therefore (ii) $\gamma = gcd(\beta, p') = gcd(d_1, d_2, p) = 1$ as well. This then implies $p'' = p'/\gamma = p\gamma = p$. We will use this fact later in Section 3.

Returning to the corollary we observe from (4) that all of the S_i are 0 except for those that are some multiple of $\frac{\beta}{\alpha}m = \beta m$. We make a further observation.

Lemma 5. *Given $x_1, x_2, \ldots, x_n$, let $\delta_1, \delta_2, \ldots, \delta_n$ be defined by*

$$f(x) = (x - x_1)(x - x_2) \cdots (x - x_n) = x^n - \delta_1 x^{n-1}$$
$$+ \cdots + (-1)^n \delta_n.$$

For $i = 1, 2, \ldots, n$, set $S_i = x_1^i + x_2^i + \cdots + x_n^i$. If there exists an integer v such that

$$\forall i, 1 \le i \le n, \text{ if } v \nmid i \text{ then } S_i = 0,$$

then

$$\forall i, 1 \le i \le n, \text{ if } v \nmid i \text{ then } \delta_i = 0.$$

Proof. We assume that $v > 1$ since otherwise the lemma is trivially correct. Now assume that if $v \nmid i$ then $S_i = 0$. We prove by induction on i that if $v \nmid i$ then $\delta_i = 0$.

Note that we can rewrite Newton's formulae (3) as

$$\delta_i = \frac{(-1)^{i+1}}{i}\left(S_i + \sum_{t=1}^{i-1}(-1)^t\delta_t S_{i-t}\right). \qquad (7)$$

For $i = 1, \delta_1 = S_1 = 0$. Suppose now that $\delta_j = 0$ for all $j < i$ such that $v \nmid j$. If $v \nmid i$ then, $\forall t < i$, at least one of $v \nmid t$ or $v \nmid (i - t)$ is true so $\delta_t S_{i-t} = 0$ and, since $S_i = 0$ we have from (7) that $\delta_i = 0$. ∎

Now (recalling from the comment following Lemma 4 that $\alpha = 1$) set $v = \frac{\beta}{\gamma}m = \beta m$. From Corollary 1 we know that if $v \nmid i$ then $S_i = 0$. Lemma 5 then implies that if $v \nmid i$ then $\delta_i = 0$. To solve for δ_i when $v|i$ we can rewrite Newton's formulae, simplifying by discarding all zero terms to get

$$0 = S_v + (-1)^v v\delta_v$$

$$0 = S_{2v} + (-1)^v\delta_v S_v + (-1)^{2v}2v\delta_{2v}$$

$$0 = S_{3v} + (-1)^v \delta_v S_{2v} + (-1)^{2v} \delta_{2v} S_v + (-1)^{3v} 3v \delta_{3v}$$

$$\vdots \quad \vdots \quad \vdots$$

$$0 = S_{(d-1)v} + (-1)^v \delta_v S_{(d-2)v} + \cdots + (-1)^{(d-2)v} \delta_{(d-2)v} S_v$$
$$+ (-1)^{(d-1)v}(d-1)v \delta_{(d-1)v}$$

$$0 = S_{dv} + (-1)^v \delta_v S_{(d-1)v} + \cdots + (-1)^{(d-1)v} \delta_{(d-1)v} S_v$$
$$+ (-1)^{dv} dv \delta_{dv},$$

where $d = \frac{d_1 \alpha}{\beta} = \frac{d_1}{\beta} = d_1'$. Now for $i = 1, 2, \ldots, d$ set $Y_i(v) = S_{iv}$ to be the known functions and $X_i(v) = \delta_{iv}$ to be the functions for which we want to solve. The system above then becomes

$$-Y_1(v) = (-1)^v v X_1(v)$$

$$-Y_2(v) = (-1)^v X_1(v) Y_1(v) + (-1)^{2v} 2v X_2(v)$$

$$-Y_3(v) = (-1)^v X_1(v) Y_2(v) + (-1)^{2v} X_2(v) Y_1(v)$$
$$+ (-1)^{3v} 3v X_3(v)$$

$$\vdots \quad \vdots \quad \vdots \qquad\qquad (8)$$

$$-Y_{(d-1)}(v) = (-1)^v X_1(v) Y_{(d-2)}(v) + \cdots$$
$$+ (-1)^{(d-1)v}(d-1)v X_{(d-1)}(v)$$

$$-Y_d(v) = (-1)^v X_1(v) Y_{(d-1)}(v) + \cdots$$
$$+ (-1)^{(d-1)v} X_{(d-1)v} Y_1(v) + (-1)^{dv} dv X_d(v),$$

which is nonsingular and can therefore be solved for $X_i(v)$ in terms of the $Y_i(v)$. In the next section we see examples of this technique.

As pointed out in the comments after Lemma 2, we do not need to know $\delta_n = X_d(v)$ to calculate $T(\vec{C}_n^{p,q})$. So we actually only need to solve for the $d-1$ functions $X_i(v) = \delta_{iv}$, $i = 1, 2, \ldots, d-1$ and not all the d functions.

Before proceeding to calculate $T(\vec{C}_n^{p,q})$, we note that the expression for $Y_i(v) = S_{iv}$ given in Equation (4) of Corollary 1 is in the form of a sum of binomial coefficients of an arithmetic series. While this looks unwieldy, we will actually be able to use the following useful lemma to derive a closed form for the sums.

Lemma 6. *Let $n \geq 0$ and let j, d satisfy $0 \leq j \leq d-1$. Then*

$$C_n^j + C_n^{d+j} + C_n^{2d+j} + \cdots + C_n^{\lfloor \frac{n-j}{d} \rfloor d + j}$$

$$= \frac{2^n}{d} \sum_{k=0}^{d-1} \cos^n \frac{k}{d} \pi \cos \frac{k(n-2j)}{d} \pi.$$

Proof. Let $\omega = e^{\frac{2\pi i}{d}}$. So ω is the dth root of unity and

$$\sum_{j=0}^{d-1} (\omega^k)^j = \begin{cases} d & k = 0 \pmod{d}, \\ 0 & \text{otherwise.} \end{cases}$$

For each k, $0 \leq k \leq d-1$, multiply both sides of the following identity by $\omega^{k(d-j)}$:

$$\sum_{l=0}^{n} C_n^l (\omega^k)^l = (1 + \omega^k)^n.$$

Summing up the d identities for $k = 0, 1, \ldots, d-1$ yields

$$d\left(C_n^j + C_n^{d+j} + C_n^{2d+j} + \cdots + C_n^{\lfloor \frac{n-j}{d} \rfloor d + j}\right)$$

$$= \sum_{k=0}^{d-1} \omega^{k(d-j)}(1 + \omega^k)^n. \quad (9)$$

Since

$$\sum_{k=0}^{d-1} \omega^{k(d-j)}(1 + \omega^k)^n = \sum_{k=0}^{d-1} e^{\frac{2k(d-j)\pi i}{d}}(1 + e^{\frac{2k\pi i}{d}})^n$$

$$= \sum_{k=0}^{d-1} e^{\frac{2k(d-j)\pi i}{d}} \left(2 \cos \frac{k}{d} \pi e^{\frac{k\pi i}{d}}\right)^n$$

$$= 2^n \sum_{k=0}^{d-1} \cos^n \frac{k}{d} \pi e^{\frac{k(n-2j)\pi i}{d}},$$

taking the real part of (9) proves the lemma. $\blacksquare$

3. THE TECHNIQUE AND EXAMPLES

In this section we use the facts developed in the previous section to derive formulas for $T(\vec{C}_n^{p,q})$ as a function of m when $n = d_1 m$ and $q = d_2 m + p$. Recall that from Lemma 4, we may assume that both $\alpha = \gcd(p, m) = 1$ and $\gamma = \gcd(d_1, d_2, p) = 1$ since, if not, $T(\vec{C}_{d_1 m}^{p, d_2 m + p}) = 0$. Furthermore, from the comments following the lemma, we may also assume that $p'' = p'/\gamma = (p/\alpha)/\gamma = p$.

3.1. The Technique

Gathering together all of the facts from the previous section yields the following four step technique.

1. Calculate $\beta = \gcd(d_1, d_2)$, $d_1' = d_1/\beta$, $d_2' = d_2/\beta$ and $\bar{d_2'}$ such that $d_2' \bar{d_2'} \equiv 1 \pmod{d_1'}$.
2. Set $v = m\beta$. For $\ell = 1, 2, \ldots, d_1' - 1$ use Corollary 1 and Lemma 6 to calculate

$$Y_\ell(v) = S_{\ell v}$$

$$= n \sum_{t=0}^{\lfloor (\ell v - x_\ell)/d_1' \rfloor} C_{\ell v}^{x_\ell + t d_1'} \qquad (10)$$

$$= n \frac{2^{\ell v}}{d_1'} \sum_{k=0}^{d_1' - 1} \left(\cos^\ell \frac{k}{d_1'} \pi\right)^v \cos \frac{k(\ell v - 2x_\ell)}{d_1'} \pi$$

$$= v 2^{\ell v} \sum_{k=0}^{d_1' - 1} \left(\cos \frac{k}{d_1'} \pi\right)^{\ell v} \cos \frac{k(\ell v - 2x_\ell)}{d_1'} \pi,$$

where $x_\ell = (-\ell p'' \bar{d_2'}) \bmod d_1' = (-\ell p \bar{d_2'}) \bmod d_1'$.

3. Use (8) to solve for $\delta_{\ell v} = X_\ell(v)$, $\ell = 1, 2, \ldots, d_1' - 1$.
4. Substitute the derived $\delta_{\ell v}$ values into (2) and use the fact that if $v \nmid i$ then $\delta_i = 0$ to derive

$$T\big(\vec{C}_n^{p,q}\big) = f'(2) = n2^{n-1}$$

$$+ \sum_{\ell=1}^{d_1'-1} (-1)^{\ell \beta m}(n - \ell \beta m)\delta_{\ell \beta m} 2^{n-\ell \beta m - 1}. \quad (11)$$

We also make two observations that can reduce the number of cases that need to be examined. The first is simply that if $\beta = \gcd(d_1, d_2)$, $d_1' = d_1/\beta$, and $d_2' = d_2/\beta$, then setting $g(m) = T(\vec{C}_{d_1' m}^{p, d_2' m + p})$ and $h(m) = T(\vec{C}_{d_1 m}^{p, d_2 m + p})$ gives $h(m) = g(\beta m)$. Since, in our technique, solving for both $g(m)$ and $h(m)$ involve the 'same amount of work', i.e., solving for $d'-1$ unknowns from $d'-1$ equations, we might as well solve for $g(m)$. For example, instead of solving for $T(\vec{C}_{6m}^{1,4m+1})$ we can solve for $T(\vec{C}_{3m}^{1,2m+1})$.

The second more interesting observation is stated next.

Lemma 7. *Let* $\beta = \gcd(d_1, d_2)$, $d_1' = d_1/\beta$ *and* $d_2' = d_2/\beta$. *If* $p_1 \equiv p_2 \pmod{d_1'}$, $\gcd(p_1, m) = \gcd(p_2, m) = 1$, *and* $\gcd(\beta, p_1) = \gcd(\beta, p_2) = 1$, *then*

$$T\big(\vec{C}_{d_1 m}^{p_1, d_2 m + p_1}\big) = T\big(\vec{C}_{d_1 m}^{p_2, d_2 m + p_2}\big).$$

Proof. Examining our technique for deriving $T(\vec{C}_{d_1 m}^{p, d_2 m + p})$, we note that the only place in which p is used is in the definition of $x_\ell = (-\ell p d_2') \pmod{d_1'}$ in Step 2. This value is the same for all ℓ if $p_1 \equiv p_2 \pmod{d_1'}$ so the proof follows. The reason for the requirement that $\gcd(p_1, m) = \gcd(p_2, m) = 1$ is that in Step 2 we were explicitly using the fact that $\gcd(p, m) = 1$ to force $p'' = p$. As seen before, if $\gcd(p, m) \neq 1$ then the graph has no spanning trees, and so this is not an interesting case. ∎

As an example, this lemma would imply that

if $2 \nmid m$ then $\qquad T(\vec{C}_{3m}^{1,2m+1}) = T(\vec{C}_{3m}^{4,2m+4})$,

if $2 \nmid m$ and $5 \nmid m$ then $\quad T(\vec{C}_{3m}^{2,2m+2}) = T(\vec{C}_{3m}^{5,2m+5})$,

for all m $\qquad\qquad T(\vec{C}_{3m}^{3,2m+3}) = T(\vec{C}_{3m}^{6,2m+6})$.

Note that if $2|m$ then $T(\vec{C}_{3m}^{4,2m+4}) = 0$, $T(\vec{C}_{3m}^{2,2m+2}) = 0$ and $T(\vec{C}_{3m}^{6,2m+6}) = 0$; if $5|m$ then $T(\vec{C}_{3m}^{5,2m+5}) = 0$; while if $3|m$ then $T(\vec{C}_{3m}^{3,2m+3}) = T(\vec{C}_{3m}^{6,2m+6}) = 0$.

3.2. Examples

We illustrate this technique by first evaluating the simplest case.

Example 1. $T(\vec{C}_{2m}^{1,m+1})$.

In this case $p = d_2 = 1$, $d_1 = 2$ so $\beta = 1$, $v = m$ and $d_1' = d_1/\beta = 2$. Since $d_1' = 2$ we only need to find

$\delta_v = X_1(v)$. Note that $\bar{d}_2' = d_2' = 1$ so $x_1 = (-1) \bmod 2 = 1$. Substituting into (10) gives

$$Y_1(v) = S_m = 2m \sum_{t=0}^{\lfloor \frac{m-1}{2} \rfloor} C_m^{2t+1} = 2m2^{m-1}. \quad (12)$$

The system of Equations (8) in this case is only the one equation

$$-Y_1(v) = (-1)^v v X_1(v)$$

or

$$-2m2^{m-1} = -Y_1(v) = (-1)^v v X_1(v) = (-1)^m m\delta_m,$$

so

$$\delta_m = (-1)^{m+1} 2^m.$$

Substituting into (11) yields

$$T\big(\vec{C}_{2m}^{1,m+1}\big) = (2m)2^{2m-1} + (-1)^m m\delta_m 2^{m-1} = m2^{2m-1}. \quad (13)$$

Example 2. $T(\vec{C}_{2m}^{2,m+2})$.

In this case $p = d_1 = 2$, $d_2 = 1$, $\beta = 1$ so $v = m$, $d_1' = d_1/\beta = 2$.

The major difference in this case is that we must note that if $\alpha = \gcd(m, p) \neq 1$, i.e., m is even, then $T(\vec{C}_{2m}^{2,m+2}) = 0$. So, for the rest of the derivation we assume that m is odd.

Since $d_1' = 2$ we only need to find $\delta_v = X_1(v)$. Note that $\bar{d}_2' = d_2' = 1$ so $x_1 = (-1 \cdot 2) \bmod 2 = 0$. Substituting into (10) gives

$$Y_1(v) = S_m = 2m \sum_{t=0}^{\lfloor \frac{m}{2} \rfloor} C_m^{2t} = 2m2^{m-1}.$$

$Y_1(v) = 2m2^{m-1}$ is exactly the same as (12) so following the same derivation as in (12)-(13) we find, that if m is odd then

$$T\big(\vec{C}_{2m}^{2,m+2}\big) = (2m)2^{2m-1} + (-1)^m m\delta_m 2^{m-1} = m2^{2m-1}.$$

Thus,

$$T\big(\vec{C}_{2m}^{2,m+2}\big) = \begin{cases} m2^{2m-1} & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

Example 3. $T(\vec{C}_{4m}^{1,m+1})$.

In this case $p = d_2 = 1$, $d_1 = 4$, $\beta = 1$ so $v = m$, $d_1' = d_1/\beta = 4$. We therefore only need to derive the three functions $\delta_{\ell v} = X_\ell(v)$, $\ell = 1, 2, 3$. As before $\bar{d}_2' = d_2' = 1$.

We therefore find $x_1 = 3$, $x_2 = 2$, $x_3 = 1$. Substituting into (10) yields

$$Y_1(v) = v\left(2^v + 2^{\frac{v}{2}+1}\cos\frac{v-6}{4}\pi\right),$$

$$Y_2(v) = v\left(2^{2v} + 2^{v+1}\cos\frac{v-2}{2}\pi\right),$$

$$Y_3(v) = v\left(2^{3v} + 2^{\frac{3v}{2}+1}\cos\frac{3v-2}{4}\pi\right).$$

The system of Equations (8) in this case is

$$-Y_1(v) = (-1)^v v X_1(v)$$

$$-Y_2(v) = (-1)^v X_1(v)Y_1(v) + (-1)^{2v}2vX_2(v)$$

$$-Y_3(v) = (-1)^v X_1(v)Y_2(v) + (-1)^{2v}X_2(v)Y_1(v)$$

$$+ (-1)^{3v}3vX_3(v),$$

and solving for $X_\ell(v)$ yields

$$X_1(v) = (-1)^v\left(-2^v + 2^{\frac{v}{2}+1}\cos\frac{v-2}{4}\pi\right),$$

$$X_2(v) = (-1)^{2v}\left(2^v - 2^{\frac{3v}{2}+1}\cos\frac{v-2}{4}\pi\right),$$

$$X_3(v) = (-1)^{3v}(-2^{2v}).$$

Equation (11) states

$$T(\vec{C}_{4m}^{1,m+1}) = 4m2^{4m-1} + (-1)^m 3m\delta_m 2^{3m-1}$$

$$+ (-1)^{2m}2m\delta_{2m}2^{2m-1} + (-1)^{3m}m\delta_{3m}2^{m-1}.$$

Substituting in the values $\delta_{\ell m} = X_\ell(m)$ and simplifying yields

$$T(\vec{C}_{4m}^{1,m+1}) = m\left(2^{4m-1} + 2^{3m-1} + 2^{\frac{7m}{2}}\cos\frac{m-2}{4}\pi\right).$$

Example 4. $T(\vec{C}_{3m}^{p,d_2 m+p})$ where $1 \leq p \leq 3$, $1 \leq d_2 \leq 2$.

We will first work through the six different (p, d) cases, stating our results and showing that there are actually only three distinct cases (with the remaining three being isomorphic to the others except, possibly, when they are equal to 0). We will then derive the formulas for the three distinct cases at the end of the section.

(i) $T(\vec{C}_{3m}^{1,m+1})$.

$$T(\vec{C}_{3m}^{1,m+1}) = m\left(2^{3m-1} + 2^{m-1} - 2^{2m}\cos\frac{m-2}{3}\pi\right).$$

(ii) $T(\vec{C}_{3m}^{1,2m+1})$.

$$T(\vec{C}_{3m}^{1,2m+1}) = m\left(2^{3m-1} + 2^{m-1} + 2^{2m}\cos\frac{m-1}{3}\pi\right).$$

(iii) $T(\vec{C}_{3m}^{3,2m+3})$.

$$T(\vec{C}_{3m}^{3,2m+3}) = \begin{cases} m\left(2^{3m-1} + 2^{m-1} - 2^{2m}\cos\frac{m}{3}\pi\right) & \text{if } 3 \nmid m, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) $T(\vec{C}_{3m}^{2,m+2})$.

When m is even $T(\vec{C}_{3m}^{2,m+2}) = 0$.

When m is odd $\vec{C}_{3m}^{2,m+2}$ is isomorphic to $\vec{C}_{3m}^{1,2m+1}$. More explicitly,

$$\vec{C}_{3m}^{1,2m+1} \Longleftrightarrow \vec{C}_{3m}^{2,m+2}$$

with the trivial vertex mapping $i \longleftrightarrow 2i$. Therefore,

$$T(\vec{C}_{3m}^{2,m+2}) = \begin{cases} m\left(2^{3m-1} + 2^{m-1} + 2^{2m}\cos\frac{m-1}{3}\pi\right) & \text{if } 2 \nmid m, \\ 0 & \text{otherwise.} \end{cases}$$

(v) $T(\vec{C}_{3m}^{2,2m+2})$.

When m is even $T(\vec{C}_{3m}^{2,2m+2}) = 0$.

When m is odd $\vec{C}_{3m}^{2,2m+2}$ is isomorphic to $\vec{C}_{3m}^{1,m+1}$ using the mapping

$$\vec{C}_{3m}^{1,m+1} \Longleftrightarrow \vec{C}_{3m}^{2,2m+2}$$

with the same trivial vertex mapping $i \longleftrightarrow 2i$.

Therefore,

$$T(\vec{C}_{3m}^{2,2m+2}) = \begin{cases} m\left(2^{3m-1} + 2^{m-1} - 2^{2m}\cos\frac{m-2}{3}\pi\right) & \text{if } 2 \nmid m, \\ 0 & \text{otherwise.} \end{cases}$$

(vi) $T(\vec{C}_{3m}^{3,m+3})$.

If $3|m$ then $T(\vec{C}_{3m}^{3,m+3}) = 0$. If $3 \nmid m$ then $\vec{C}_{3m}^{3,m+3}$ is isomorphic to $\vec{C}_{3m}^{3,2m+3}$. To see this note that if $3 \nmid m$ then $\gcd(3m, m+3) = 1$ so there exists $x < 3m$ such that $x(m+3) \equiv 1 \pmod{3m}$. We can then define the isomorphism

$$\vec{C}_{3m}^{3,m+3} \Longleftrightarrow \vec{C}_{3m}^{3,2m+3}$$

using the vertex mapping $i \longleftrightarrow ix(2m+3)$. To see this is an isomorphism let

$$f(i) = ix(2m+3) \pmod{3m}$$

$$= ix(m+3+m) \pmod{3m}$$

$$= i(1+xm) \pmod{3m}.$$

Then

$$f(i+3) = (i+3)(1+xm) \pmod{3m}$$

$$= f(i) + 3 \pmod{3m}$$

and

$$f(i+m+3) = (i+m+3)x(2m+3) \pmod{3m}$$

$$= ix(2m+3) + (m+3)x(2m+3) \pmod{3m}$$

$$= f(i) + 2m + 3 \pmod{3m}.$$

So, if $(i, i+3) \in \vec{C}_{3m}^{3,m+3}$ then $(f(i), f(i+3)) \in \vec{C}_{3m}^{3,2m+3}$ and if $(i, i+m+3) \in \vec{C}_{3m}^{3,m+3}$, then $(f(i), f(i+m+3)) \in \vec{C}_{3m}^{3,2m+3}$.

Since $f(i)$ is a one-one function from $[0, 3m-1]$ into itself we have exhibited an isomorphism.

Note that if $3|m$, then $T(\vec{C}_{3m}^{3,2m+3}) = 0$. We have thus proved

$$T(\vec{C}_{3m}^{3,m+3}) = T(\vec{C}_{3m}^{3,2m+3})$$

$$= \begin{cases} m\left(2^{3m-1} + 2^{m-1} - 2^{2m}\cos\frac{m}{3}\pi\right) & \text{if } 3\nmid m, \\ 0 & \text{otherwise.} \end{cases}$$

Derivations of (i), (ii), (iii), (iv), (v), (vi). In (a), (b), and (c) below we derive the formulas for the numbers of spanning trees in the above six graphs to verify the validity of our claim.

(a) $T(\vec{C}_{3m}^{2,2m+2})$ and $T(\vec{C}_{3m}^{1,m+1})$. ((i) and (v))

We already saw that when m is even $T(\vec{C}_{3m}^{2,2m+2}) = 0$. We also saw that when m is odd $\vec{C}_{3m}^{2,2m+2}$ is isomorphic to $\vec{C}_{3m}^{1,m+1}$. We therefore only have to evaluate the number of spanning trees for $\vec{C}_{3m}^{1,m+1}$.

In this case $p = 1$, $d_1 = 3$, $d_2 = 1$, $\beta = 1$ so $v = m$, $d_1' = d_1/\beta = 3$. We only need to derive the two functions $\delta_{\ell v} = X_\ell(v), \ell = 1, 2$. Now $d_2' = d_2/\beta = 1$, $\bar{d}_2' = 2$. We therefore find $x_1 = 1$, $x_2 = 2$. Substituting into (10) yields

$$Y_1(v) = v\left(2^v + 2\cos\frac{v-2}{3}\pi\right),$$

$$Y_2(v) = v\left(2^{2v} + 2(-1)^v\cos\frac{v-2}{3}\pi\right).$$

The system of Equations (8) in this case is

$$-Y_1(v) = (-1)^v v X_1(v),$$

$$-Y_2(v) = (-1)^v X_1(v)Y_1(v) + (-1)^{2v} 2v X_2(v)$$

and solving for $X_\ell(v)$ yields

$$X_1(v) = -(-1)^v\left(2^v + 2\cos\frac{v-2}{3}\pi\right),$$

$$X_2(v) = (2^{v+1} - (-1)^v)\cos\frac{v-2}{3}\pi + 2\cos^2\frac{v-2}{3}\pi.$$

Equation (11) states

$$T(\vec{C}_{3m}^{1,m+1}) = 3m2^{3m-1} + (-1)^m 2m\delta_m 2^{2m-1}$$
$$+ (-1)^{2m} m\delta_{2m} 2^{m-1}.$$

Substituting in the values $\delta_{\ell m} = X_\ell(m)$ and simplifying yields

$$T(\vec{C}_{3m}^{1,m+1}) = m\left(2^{3m-1} + 2^{m-1} - 2^{2m}\cos\frac{m-2}{3}\pi\right),$$

proving **(i)** and **(v)**.

(b) $T(\vec{C}_{3m}^{1,2m+1})$ and $T(\vec{C}_{3m}^{2,m+2})$ ((ii) and (iv))

We already saw that when m is even $T(\vec{C}_{3m}^{2,m+2}) = 0$ and, when m is odd, $\vec{C}_{3m}^{2,m+2}$ is isomorphic to $\vec{C}_{3m}^{1,2m+1}$. We therefore only have to calculate the number of spanning trees in

$\vec{C}_{3m}^{1,2m+1}$. In this case $p = 1$, $d_1 = 3$, $d_2 = 2$, $\beta = 1$ so $v = m$, $d_1' = d_1/\beta = 3$. We therefore only need to derive the two functions $\delta_{\ell v} = X_\ell(v), \ell = 1, 2$. Now $d_2' = d_2/\beta = 2$, $\bar{d}_2' = d_2' = 2$. We find $x_1 = 2$, $x_2 = 1$. Substituting into (10) yields

$$Y_1(v) = v\left(2^v + 2\cos\frac{v-4}{3}\pi\right),$$

$$Y_2(v) = v\left(2^{2v} + 2(-1)^v\cos\frac{v-4}{3}\pi\right).$$

The system of Equations (8) in this case is again

$$-Y_1(v) = (-1)^v v X_1(v),$$

$$-Y_2(v) = (-1)^v X_1(v)Y_1(v) + (-1)^{2v} 2v X_2(v)$$

and solving for $X_\ell(v)$ yields

$$X_1(v) = -(-1)^v\left(2^v + 2\cos\frac{v-4}{3}\pi\right),$$

$$X_2(v) = (2^{v+1} - (-1)^v)\cos\frac{v-4}{3}\pi + 2\cos^2\frac{v-4}{3}\pi.$$

Equation (11) states

$$T(\vec{C}_{3m}^{1,2m+1}) = 3m2^{3m-1} + (-1)^m 2m\delta_m 2^{2m-1}$$
$$+ (-1)^{2m} m\delta_{2m} 2^{m-1}.$$

Substituting in the values $\delta_{\ell m} = X_\ell(m)$ and simplifying yields

$$T(\vec{C}_{3m}^{1,2m+1}) = m\left(2^{3m-1} + 2^{m-1} - 2^{2m}\cos\frac{m-4}{3}\pi\right)$$

$$= m\left(2^{3m-1} + 2^{m-1} + 2^{2m}\cos\frac{m-1}{3}\pi\right)$$

where the last equality comes from the fact $\cos(x - \pi) = -\cos x$. This proves **(ii)** and **(iv)**.

(c) $T(\vec{C}_{3m}^{3,2m+3})$ and $T(\vec{C}_{3m}^{3,m+3})$ ((iii) and (vi))

We already saw that when $3|m$ both graphs have no spanning trees and when $3 \nmid m$ the two graphs are isomorphic. We therefore only consider $T(\vec{C}_{3m}^{3,2m+3})$. In this case $p = 3$, $d_1 = 3$, $d_2 = 2$, $\beta = 1$ so $v = m$, $d_1' = d_1/\beta = 3$. We therefore only need to derive the two functions $\delta_{\ell v} = X_\ell(v), \ell = 1, 2$. Now $d_2' = d_2/\beta = 2$, $\bar{d}_2' = d_2' = 2$. We find $x_1 = 0$, $x_2 = 0$. Substituting into (10) yields

$$Y_1(v) = v\left(2^v + 2\cos\frac{v}{3}\pi\right),$$

$$Y_2(v) = v\left(2^{2v} + 2(-1)^v\cos\frac{v}{3}\pi\right)$$

The system of Equations (8) in this case is yet again

$$-Y_1(v) = (-1)^v v X_1(v),$$

$$-Y_2(v) = (-1)^v X_1(v)Y_1(v) + (-1)^{2v} 2v X_2(v)$$

and solving for $X_\ell(v)$ yields

$$X_1(v) = -(-1)^v \left(2^v + 2 \cos \frac{v}{3}\pi \right),$$

$$X_2(v) = (2^{v+1} - (-1)^v) \cos \frac{v}{3}\pi + 2 \cos^2 \frac{v}{3}\pi.$$

Equation (11) states

$$T\big(\vec{C}_{3m}^{3,2m+3}\big) = 3m2^{3m-1} + (-1)^m 2m\delta_m 2^{2m-1}$$
$$+ (-1)^{2m} m\delta_{2m} 2^{m-1}.$$

Substituting in the values $\delta_{\ell m} = X_\ell(m)$ and simplifying yields (if $3 \nmid m$)

$$T\big(\vec{C}_{3m}^{3,2m+3}\big) = m \left(2^{3m-1} + 2^{m-1} - 2^{2m} \cos \frac{m}{3}\pi \right),$$

proving **(iii)** and **(vi)**.

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