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# On a conjecture of Fiedler and Markham

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**Abstract**

For the Hadamard product $A \circ A^{-1}$ of an $M$-matrix $A$ and its inverse $A^{-1}$, Fiedler and Markham conjectured that $q(A \circ A^{-1}) \geqslant 2/n$ (see M. Fiedler and T.L. Markham, Linear Algebra Appl. 101(1988) 1–8), where $q(A \circ A^{-1})$ is the smallest eigenvalue (in modulus) of $A \circ A^{-1}$. The present paper studies this conjecture (an incorrect proof is given in Li Ching and Chen Ji-cheng, Linear Algebra Appl. 144 (1991) 171–178), and establishes $q(A \circ A^{-1}) > (2/n)((n-1)/n)$. For some special matrices, the conjecture is proved. 

## 1. Introduction

The Hadamard product of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size is defined as the matrix $C = A \circ B = (a_{ij}b_{ij})$.

All $M$-matrices considered here are nonsingular $M$-matrices. If $A$ is an $M$-matrix, there exists a positive eigenvalue of $A$ equal to $(\rho(A^{-1}))^{-1}$, in which $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix $A^{-1}$. We denote this eigenvalue by $q(A)$. By Johnson [1] and Fiedler and Markham [2] we know that the Hadamard product $A \circ B^{-1}$ of an $n \times n$ $M$-matrix $A$ and the inverse of an $n \times n$ $M$-matrix $B$ is again an $M$-matrix. For $B = A$, Fiedler and Markham [2] posed the following conjecture,

$$q(A \circ A^{-1}) \geqslant \frac{2}{n}.$$

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We note that this conjecture was considered in Ref. [3], and the authors stated that they gave a proof for this conjecture. But, we have found a mistake in their proof. Indeed, the following inequality

$$\frac{a_{j_1j_2} + a_{j_2j_2}b_{ij_2}/b_{ij_1}}{a_{j_1j_2} + (a_{j_2j_2} - 1)} b_{ij_1} \leqslant \frac{a_{j_2j_2}}{a_{j_2j_2} - 1} b_{ij_2}$$

does not hold for $a_{j_1j_2} < 0, a_{j_1j_2} + (a_{j_2j_2} - 1) > 0, b_{ij_2} > b_{ij_1} > 0$. For details, see the proof of Theorem 1 in Ref. [3]. The aim of this paper is to consider this conjecture. We shall show the following inequality

$$q(A \circ A^{-1}) > \frac{2}{n}\frac{n-1}{n}$$

and in some cases we provide that

$$q(A \circ A^{-1}) \geqslant \frac{2}{n}, \quad n > 3.$$

We need some basic results to prove the above two inequalities.

## 2. Some lemmas

**Lemma 2.1.** *If $P$ is an irreducible M-matrix and $Pz \geqslant kz$ for a nonnegative nonzero vector $z$, then $k \leqslant q(P)$.*

**Proof.** This is the Proposition 1 of Ref. [2]. □

**Lemma 2.2.** *Let $A = (a_{ij})$ be a strictly diagonally dominant matrix by row and column, i.e.,*

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \qquad |a_{ii}| > \sum_{j \neq i} |a_{ji}|,$$

*for all $i$. Then for $A^{-1} = (b_{ij})$, we have:*

(a) $|b_{ji}| \leqslant \frac{\sum_{t \neq j} |a_{jt}|}{|a_{jj}|} |b_{ii}|$.

(b) $|b_{ij}| \leqslant \frac{\sum_{t \neq j} |a_{tj}|}{|a_{jj}|} |b_{ii}|$.

**Proof.** Let

$$r_i = \frac{\sum_{t \neq i} |a_{it}| + \epsilon}{|a_{ii}|},$$

where $\epsilon$ is sufficiently small such that $0 < r_i < 1$ for all $i$, and let $D_i = \mathrm{diag}(r_1, \ldots, r_{i-1}, 1, r_{i+1}, \ldots, r_n)$. Then it is easy to see that the matrix $AD_i$ is again strictly diagonally dominant by row. Therefore, by virtue of the Proposition 2 of Ref. [2], we can deduce readily that

$$\frac{1}{r_j}|b_{ji}| < |b_{ii}|,$$

for all $j, j \neq i$. This gives the following when $\epsilon \to 0^+$,

$$|b_{ji}| \leqslant \frac{\sum_{t \neq j} |a_{jt}|}{|a_{jj}|} |b_{ii}|,$$

for all $j, j \neq i$. Now when $i$ is varying from 1 to $n$, it yields the assertion (a).

Analogically, letting

$$c_i = \frac{\sum_{t \neq i} |a_{ti}| + \epsilon}{|a_{ii}|},$$

and considering the matrix $F_iA$, where $F_i = \mathrm{diag}(c_1, \ldots, c_{i-1}, 1, c_{i+1}, \ldots, c_n)$, we may conclude that

$$|b_{ij}| \leqslant \frac{\sum_{t \neq j} |a_{tj}|}{|a_{jj}|} |b_{ii}|,$$

for all $i, j, i \neq j$. □

**Lemma 2.3.** *Let $A = (a_{ij})$ be an $n \times n$ M-matrix, and let $A^{-1} = (b_{ij})$ be doubly stochastic, then*

$$Ae = A^{\mathrm{T}}e = e,$$

where $e = (1, 1, \ldots, 1)^{\mathrm{T}}$.

**Proof.** This lemma is obvious, see Ref. [3]. □

**Lemma 2.4.** *Let $A = (a_{ij})$ be an $n \times n$ M-matrix, and let the inverse $A^{-1} = (b_{ij})$ be doubly stochastic, then*

(c) $b_{ii}a_{ii} \geqslant 1$,

(d) $b_{ii} \geqslant \dfrac{1}{n - \sum_{j \neq i} \frac{1}{a_{jj}}}$.

**Proof.** (c) Let

$$A = \begin{pmatrix} a_{11} & b^{\mathrm{T}} \\ c & A_1 \end{pmatrix}.$$

Then by (Berman and Plemmons [4], Ch. 6), we have $A_1^{-1} \geqslant 0$, $b^{\mathrm{T}} \leqslant 0$, $c \leqslant 0$. Hence,

$$b_{11}a_{11} = \frac{|A_1|a_{11}}{|A|} \geqslant \frac{|A_1|(a_{11} - b^{\mathrm{T}}A_1^{-1}c)}{|A|} = 1.$$

The rest follows similarly. This completes the proof of (c).

Now, we consider (d). In the light of Lemmas 2.2 and 2.3, we may infer

$$b_{ji} \leqslant \frac{a_{jj} - 1}{a_{jj}} b_{ii},$$

for all $i, j$, $i \neq j$. Therefore,

$$1 = b_{ii} + \sum_{j \neq i} b_{ji} \leqslant b_{ii} + \sum_{j \neq i}\left(1 - \frac{1}{a_{jj}}\right) b_{ii} = \left(n - \sum_{j \neq i} \frac{1}{a_{jj}}\right) b_{ii},$$

i.e.,

$$b_{ii} \geqslant \frac{1}{n - \sum_{j \neq i} \frac{1}{a_{jj}}}. \qquad \square$$

**Lemma 2.5.** *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & A_1 & \\ a_{n1} & & & \end{pmatrix}$$

*be either strictly diagonally dominant (by columns) or irreducibly diagonally dominant matrix (by columns), then*

$$(|a_{12}|, \ldots, |a_{1n}|)\bar{A}_1^{-1} \leqslant e,$$

*where* $\bar{A}_1$ *is the comparison matrix of* $A_1$ [5], $e^{\mathrm{T}} = (1, 1, \ldots, 1)$.

**Proof.** By Householder [6] $\bar{A}_1$ is an $M$-matrix. Let $(|a_{12}|, \ldots, |a_{1n}|)\bar{A}_1^{-1} = y^{\mathrm{T}} = (y_2, \ldots, y_n)^{\mathrm{T}}$, and $y_i = \max y_j$, then we have

$$\bar{A}_1^{\mathrm{T}} y = \begin{pmatrix} |a_{12}| \\ \vdots \\ |a_{1n}| \end{pmatrix}.$$

This gives

$$|a_{1i}| = |a_{ii}|y_i - \sum_{j\neq i}|a_{ji}|y_j \geqslant \left(|a_{ii}| - \sum_{j\neq i}|a_{ji}|\right)y_i,$$

i.e.,

$$y_i \leqslant \frac{|a_{1i}|}{|a_{ii}| - \sum_{j\neq i}|a_{ji}|} \leqslant 1.$$

This yields $y \leqslant e$. □

## 3. The main results

**Theorem 3.1.** *Let $A = (a_{ij})$ be an irreducible $n \times n$ M-matrix and $A^{-1} = B = (b_{ij})$ be a doubly stochastic matrix. Then*

$$q(A \circ A^{-1}) > \frac{2}{n}\frac{n-1}{n}.$$

**Proof.** By Lemma 2.3, we have for all $i$ that

(1) $a_{ii} > 1$,

(2) $a_{ii} = \sum_{j\neq i}|a_{ji}| + 1$,

For the vector $z = (a_{11}/(a_{11}-1), a_{22}/(a_{22}-1), \ldots, a_{nn}/(a_{nn}-1))^T$, let $f_i$ be the $i$th component of the vector $z^T(A \circ A^{-1})$, then by Lemma 2.2

$$\begin{aligned} f_i &= a_{ii}b_{ii}\frac{a_{ii}}{a_{ii}-1} - \sum_{j\neq i}|a_{ji}|b_{ji}\frac{a_{jj}}{a_{jj}-1} \\ &\geqslant a_{ii}b_{ii}\frac{a_{ii}}{a_{ii}-1} - \sum_{j\neq i}|a_{ji}|\frac{a_{jj}-1}{a_{jj}}b_{ii}\frac{a_{jj}}{a_{jj}-1} \\ &= b_{ii}\left(a_{ii}\frac{a_{ii}}{a_{ii}-1} - \sum_{j\neq i}|a_{ji}|\right). \end{aligned}$$

If $b_{ii} \geqslant 2/n$, it is easy to see that

$$f_i \geqslant b_{ii}\left(a_{ii} - \frac{a_{ii}-1}{a_{ii}}\sum_{j\neq i}|a_{ji}|\right)\frac{a_{ii}}{a_{ii}-1} > \frac{2}{n}\frac{a_{ii}}{a_{ii}-1}.$$

Now, let $b_{ii} < \frac{2}{n}$. Then by (c) of Lemma 2.4, we have

$$a_{ii} \geqslant \frac{1}{b_{ii}} > \frac{n}{2}.$$

Therefore, using (d) of Lemma 2.4, we have that

$$\begin{aligned} f_i &\geqslant b_{ii}\left(a_{ii} - \frac{1}{a_{ii}}(a_{ii}-1)^2\right)\frac{a_{ii}}{a_{ii}-1} \\ &\geqslant \frac{1}{n - \sum_{j\neq i}\frac{1}{a_{jj}}}\left(a_{ii} - \frac{(a_{ii}-1)^2}{a_{ii}}\right)\frac{a_{ii}}{a_{ii}-1} \\ &= \frac{1}{n - \sum_{j\neq i}\frac{1}{a_{jj}}}\left(2 - \frac{1}{a_{ii}}\right)\frac{a_{ii}}{a_{ii}-1} > \frac{2}{n}\frac{n-1}{n}\frac{a_{ii}}{a_{ii}-1}. \end{aligned}$$

Consequently, combining the above two cases, we get

$$z^{\mathrm T}(A \circ A^{-1}) > \frac{2}{n}\frac{n-1}{n} z^{\mathrm T},$$

which, when considering $(A \circ A^{-1})^{\mathrm T} z$ and using Lemma 2.1, implies

$$q(A \circ A^{-1}) > \frac{2}{n}\frac{n-1}{n}. \qquad \square$$

**Theorem 3.2.** *Let $A$ be an $n \times n$ M-matrix, $n > 2$, then*

$$q(A \circ A^{-1}) > \frac{2}{n}\frac{n-1}{n}.$$

**Proof.** For the case that $A$ is irreducible, we infer that $A^{-1}$ is positive, and $A \circ A^{-1}$ is again irreducible. By Sinkhorn [7], there exist two positive diagonal matrices $D_1$ and $D_2$ such that $D_1 A^{-1} D_2$ is doubly stochastic. Since the matrix $B = D_2^{-1} A D_1^{-1}$ is again an $M$-matrix, and

$$q(A \circ A^{-1}) = q(D_2 D_1^{-1}(B \circ B^{-1})(D_2 D_1^{-1})^{-1}) = q(B \circ B^{-1}),$$

we conclude that $q(A \circ A^{-1}) > (2/n)(n-1)/n$ in terms of Theorem 3.1.

Now let $A$ be reducible, then, without loss of generality, we may assume that $A$ has a block upper triangular form $(A_{ij})$ with irreducible diagonal blocks $A_{ii}$, where $A_{ij}$ are $n_i \times n_j$ matrices, $i, j = 1, 2, \ldots, t$. This implies that the inverse $A^{-1}$ is again block upper triangular with diagonal positive blocks $A_{ii}^{-1} > 0$, $i = 1, 2, \ldots, t$. Then since we have

$$q(A \circ A^{-1}) = \min q(A_{ii} \circ A_{ii}^{-1}),$$

and

$$q(A_{ii} \circ A_{ii}^{-1}) > \frac{2}{n_i}\frac{n_i - 1}{n_i}, \quad n_i < n,$$

it produces

$$q(A \circ A^{-1}) > \frac{2}{n}\frac{n-1}{n},$$

which completes the proof of Theorem 3.2. □

**Theorem 3.3.** *Let the $n \times n$ M-matrix $A = (a_{ij})$ be either diagonally dominant by column, or irreducibly diagonally dominant by column. Further, if $A$ is also a sum-symmetric matrix* [8] *(i.e., $\sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} a_{ji}$, for all $j$), and for all $i, j, i \neq j$*

$$|a_{ij}| \geqslant \alpha |a_{ji}|,$$

*then $q(A \circ A^{-1}) \geqslant \alpha$, where $0 < \alpha \leqslant 1$.*

**Proof.** For the vector $e = (1, 1, \ldots, 1)^{\mathrm{T}}$, we consider $(A \circ A^{-1})e$. Let $(A \circ A^{-1})e = (f_1, f_2, \ldots, f_n)^{\mathrm{T}}$, then $f_i$ is the $i$th main diagonal entry of the matrix $AA^{-T}$. Given $C = AA^{-T} = (c_{ij})$, we consider $\min f_i$. Without loss of generality, we may assume that $f_1 = \min f_i$. Then since $AC^{\mathrm{T}} = A^{\mathrm{T}}$, we have the following from Cramer's rule:

$$c_{11} = \frac{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & & & \\ \vdots & & A_1 & \\ a_{1n} & & & \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & A_1 & \\ a_{n1} & & & \end{vmatrix}} = \frac{a_{11} - (a_{12}, \ldots, a_{1n})A_1^{-1}\begin{pmatrix} a_{12} \\ \vdots \\ a_{1n} \end{pmatrix}}{a_{11} - (a_{12}, \ldots, a_{1n})A_1^{-1}\begin{pmatrix} a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}},$$

where $A_1$ is the last $(n-1) \times (n-1)$ principal submatrix of matrix $A$. If $\alpha = 1$, then $A$ is symmetric due to the sum-symmetricity, and we have that $c_{ii} = f_{ii} = 1$ for all $i$. This gives $(A \circ A^{-1})e = e$, and $q(A \circ A^{-1}) = 1$. This is a result of Ref. [9].

For $0 < \alpha < 1$, since $|a_{ij}| \geqslant \alpha |a_{ji}|$ and by Lemma 2.5, we have

$$a_{11} - (a_{12}, \ldots, a_{1n})A_1^{-1}\begin{pmatrix} \dfrac{a_{12} - \alpha a_{21}}{1-\alpha} \\ \vdots \\ \dfrac{a_{1n} - \alpha a_{n1}}{1-\alpha} \end{pmatrix} \geqslant a_{11} - \sum_{j \neq 1} |a_{j1}| \geqslant 0.$$

On the other hand, the above inequality is equivalent to

$$c_{11} = \frac{a_{11} - (a_{12}, \ldots, a_{1n})A_1^{-1}\begin{pmatrix} a_{12} \\ \vdots \\ a_{1n} \end{pmatrix}}{a_{11} - (a_{12}, \ldots, a_{1n})A_1^{-1}\begin{pmatrix} a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}} \geqslant \alpha.$$

Therefore, we get

$$(A \circ A^{-1})e \geqslant \alpha e,$$

which gives the following (by Lemma 2.1)

$$q(A \circ A^{-1}) \geqslant \alpha. \qquad \square$$

**Theorem 3.4.** *Let $A = (a_{ij})$ be an irreducible $n \times n$ M-matrix with $a_{11} = a_{22} = \cdots = a_{nn}$, $n > 2$, and for $e^{\mathrm{T}} = (1, 1, \ldots, 1)$, we have $Ae = A^{\mathrm{T}}e = ke$, $k > 0$. Then*

$$q(A \circ A^{-1}) \geqslant \frac{2}{n}.$$

**Proof.** Without loss of generality, we may assume that $k = 1$. Then $A^{-1}$ is doubly stochastic, and $A$ satisfies the conditions of Theorem 3.1. By the proof of that theorem, we have

$$f_i \geqslant \frac{1}{n - \sum_{j \neq i} \frac{1}{a_{ii}}}\left(2 - \frac{1}{a_{ii}}\right),$$

where $e^{\mathrm{T}}(A \circ A^{-1}) = (f_1, f_2, \ldots, f_n)$. Now, since $a_{11} = a_{22} = \cdots = a_{nn}$, it implies that

$$f_i \geqslant \frac{2}{n}, \quad i = 1, 2, \ldots, n,$$

that is,

$$(A \circ A^{-1})e \geqslant \frac{2}{n}e.$$

Therefore, by Lemma 2.1 we conclude that

$$q(A \circ A^{-1}) \geqslant \frac{2}{n}. \qquad \square$$

**Remark 3.1.** A special case of Theorem 3.4 yields both the examples in Ref. [2, III] and in Ref. [3].

**Remark 3.2.** For Lemma 2.2, when we repeat the procedure of its proving, we may achieve better inequalities than (a) and (b). This leads to an improvement of (d) in Lemma 2.4, and therefore, the lower bound for $q(A \circ A^{-1})$ could be more precise.

**Remark 3.3.** The conjecture is still open.

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