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Proof of a conjecture of Fiedler and Markham ☆

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Abstract

Let A be an $n \times n$ nonsingular M-matrix. For the Hadamard product $A \circ A^{-1}$, M. Fiedler and T.L. Markham conjectured in [Linear Algebra Appl. 10l (1988) 1] that $q(A \circ A^{-1}) \geqslant 2/n$, where $q(A \circ A^{-1})$ is the smallest eigenvalue (in modulus) of $A \circ A^{-1}$. We considered this conjecture in [Linear Algebra Appl. 288 (1999) 259] having observed an incorrect proof in [Linear Algebra Appl. 144 (1991) 171] and obtained that $q(A \circ A^{-1}) \geqslant (2/n)(n-1)/n$. The present paper gives a proof for this conjecture. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: M-matrix; Eigenvalues; Hadamard product; Stochastic matrix

1. Introduction

For notations and definitions not defined here we refer to [1,2].

All M-matrices considered here will be nonsingular M-matrices. For an M-matrix A, the Hadamard product $A \circ A^{-1}$ is again an M-matrix [4]. Fiedler and Markham proved [1] that $q(A \circ A^{-1}) \geqslant 1/n$, and proposed the following conjecture:

$$q(A \circ A^{-1}) \geqslant 2/n,$$

where $q(A \circ A^{-1})$ is the smallest eigenvalue (in modulus) of $A \circ A^{-1}$. We considered this conjecture [2] having observed an incorrect proof [3] and obtained that

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$q(A \circ A^{-1}) > (2/n)(n-1)/n$. The present paper is intended to prove this conjecture.

About the conjecture, I would like to point out that it is also stated in Horn and Johnson's book [6, p. 375]. The correct proof was, actually, given by the author in 1996 and the extended abstract is seen in [7].

Let t denote the transpose of a matrix or a vector.

2. Proof of the conjecture

The following lemma is Lemma 2.1 in [2].

Lemma 1. *Let $A = (a_{ij})$ be a strictly diagonally dominant matrix by row and column, i.e.,*

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad |a_{ii}| > \sum_{j \neq i} |a_{ji}|$$

for all i. Then for $A^{-1} = (b_{ij})$, we have:

(a) $|b_{ji}| \leqslant \frac{\sum_{t \neq j} |a_{jt}|}{|a_{jj}|} |b_{ii}|$,

(b) $|b_{ij}| \leqslant \frac{\sum_{t \neq j} |a_{tj}|}{|a_{jj}|} |b_{ii}|$.

Lemma 2. *Let $A = (a_{ij})$ be an $n \times n$ irreducible M-matrix, and*

$$a_{ii} = \sum_{j \neq i} |a_{ij}| + 1 = \sum_{j \neq i} |a_{ji}| + 1, \quad i = 1, 2, \ldots, n.$$

Let A_{ii} be the cofactor matrix of a_{ii}. Then

$$a_{ii} - \alpha_i^{\mathrm{t}} A_{ii}^{-1} \beta_i \leqslant n - \frac{n-1}{a_{ii}}, \quad i = 1, 2, \ldots, n,$$

where

$$\alpha_i^{\mathrm{t}} = (a_{i1}, \ldots, a_{ii-1}, a_{ii+1}, \ldots, a_{in}),$$

$$\beta_i = (a_{1i}, \ldots, a_{i-1i}, a_{i+1i}, \ldots, a_{ni})^{\mathrm{t}}.$$

Proof. We need only prove the claims for $i = 1$. The rest follows similarly. From the hypotheses, we have

$$A = \begin{pmatrix} a_{11} & \alpha_1^{\mathrm{t}} \\ \beta_1 & A_{11} \end{pmatrix}.$$

and $\alpha_1^{\mathrm{t}} = e_{n-1}^{\mathrm{t}}(I - A_{11})$, $\beta_1 = (I - A_{11})e_{n-1}$, where e_{n-1} is the $n-1$ dimensional vector of 1's. Then

$$\begin{aligned} \alpha_1^{\mathrm{t}} A_{11}^{-1} \beta_1 &= e_{n-1}^{\mathrm{t}}(I - A_{11}) A_{11}^{-1} (I - A_{11}) e_{n-1} \\ &= e_{n-1}^{\mathrm{t}} A_{11} e_{n-1} + e_{n-1}^{\mathrm{t}} A_{11}^{-1} e_{n-1} - 2(n-1). \end{aligned} \tag{1}$$

From [5], A_{11} is again an M-matrix. So $A_{11}^{-1}e_{n-1} = y = (y_2, \dots, y_n)^{\mathrm{t}} > 0$. Let $y_i = \min y_j$. Then, since $e_{n-1} = A_{11}y$, we have

$$\begin{aligned} 1 &= a_{ii}y_i - \sum_{j\neq i, j\geqslant 2} |a_{ij}|\ y_j \\ &\leqslant \left(a_{ii} - \sum_{j\neq i, j\geqslant 2} |a_{ij}|\right) y_i \\ &\leqslant (1 + \max\nolimits_{2\leqslant j\leqslant n}|a_{j1}|)y_i. \end{aligned}$$

This yields

$$\begin{aligned} e_{n-1}A_{11}^{-1}e_{n-1} &\geqslant \frac{e_{n-1}^{\mathrm{t}}e_{n-1}}{1 + \max_{2\leqslant j\leqslant n}|a_{j1}|} \\ &= \frac{n-1}{1 + \max_{2\leqslant j\leqslant n}|a_{j1}|}. \end{aligned}$$

Now,

$$e_{n-1}^{\mathrm{t}}A_{11}e_{n-1} = n - 1 + \sum_{j=2} |a_{j1}|,$$

$$a_{11} = 1 + \sum_{j=2} |a_{j1}| \geqslant 1 + \max\nolimits_{2\leqslant j\leqslant n}|a_{j1}|.$$

So from (1), it leads to

$$a_{11} - \alpha_1^{\mathrm{t}}A_{11}^{-1}\beta_1 \leqslant n - \frac{n-1}{a_{11}}.$$

The proof is completed. □

We now turn to prove the conjecture.

Theorem 3. *Let $A = (a_{ij})$ be an $n \times n$ M-matrix. Then*

$$q(A \circ A^{-1}) \geqslant \frac{2}{n}, \quad n \geqslant 2,$$

where $q(A \circ A^{-1})$ is the smallest eigenvalue (in modulus) of $A \circ A^{-1}$.

Proof. From [1] or Theorem 3.2 in [2], we may assume that A^{-1} is a doubly stochastic matrix, and $A^{-1} = B = (b_{ij}) > 0$. Then A is irreducible, and we have

$$a_{ii} > 1, \tag{2}$$

$$a_{ii} = \sum_{j\neq i} |a_{ij}| + 1 = \sum_{j\neq i} |a_{ji}| + 1 \quad \text{for all } i. \tag{3}$$

For the positive vector

$$z=\left(\frac{a_{11}}{a_{11}-1},\frac{a_{22}}{a_{22}-1},\ldots,\frac{a_{nn}}{a_{nn}-1}\right)^{\mathrm{T}},$$

let f_i be the ith component of the vector $z^{\mathrm{T}}(A\circ A^{-1})$. Then by Lemma 2

$$\begin{aligned}f_i &= a_{ii}b_{ii}\frac{a_{ii}}{a_{ii}-1}-\sum_{j\neq i}|a_{ji}|b_{ji}\frac{a_{jj}}{a_{jj}-1}\\ &\geqslant a_{ii}b_{ii}\frac{a_{ii}}{a_{ii}-1}-\sum_{j\neq i}|a_{ji}|\frac{a_{jj}-1}{a_{jj}}b_{ii}\frac{a_{jj}}{a_{jj}-1}\\ &= b_{ii}\left(a_{ii}-\frac{(a_{ii}-1)^2}{a_{ii}}\right)\frac{a_{ii}}{a_{ii}-1}\\ &= b_{ii}\left(2-\frac{1}{a_{ii}}\right)\frac{a_{ii}}{a_{ii}-1}.\end{aligned}$$

On the other hand,

$$b_{ii}=\frac{\det A_{ii}}{\det A}=\frac{1}{a_{ii}-\alpha_i^{\mathrm{t}}A_{ii}^{-1}\beta_i}.$$

So from Lemma 2, we have

$$b_{ii}\geqslant\frac{1}{n-(n-1)/a_{ii}}\quad\text{for all } i.$$

Noting (2), this gives rise to

$$\begin{aligned}f_i &\geqslant b_{ii}\left(2-\frac{1}{a_{ii}}\right)\frac{a_{ii}}{a_{ii}-1}\\ &\geqslant\frac{2-(1/a_{ii})}{n-((n-1)/a_{ii})}\frac{a_{ii}}{a_{ii}-1}\\ &\geqslant\frac{2}{n}\frac{a_{ii}}{a_{ii}-1}.\end{aligned}$$

This implies that

$$z^{\mathrm{t}}(A\circ A^{-1})\geqslant\frac{2}{n}z^{\mathrm{t}},$$

which gives the following:

$$q(A\circ A^{-1})\geqslant\frac{2}{n}.$$

The proof of the conjecture is then completed. □

Remark. The lower bound of the conjecture is best possible. See the example in [1] or [6, p. 374].

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