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On the distribution of eigenvalues of a simple undirected graph ☆

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Abstract

For a simple, undirected graph G_n, let $\lambda_i(G_n)$ be the ith largest eigenvalue of G_n. This paper presents mainly the following:

1. For $n \geqslant 4$, if G_n is incomplete, then

$$-\frac{n}{2} \leqslant \lambda_n(G_n) < -\frac{1+\sqrt{1+4\frac{n-3}{n-1}}}{2}.$$

2. Seven sufficient and necessary conditions such that $\lambda_2(G_n) = -1$.
3. $\lambda_3(G_n) = -1$ implies that $\lambda_j(G_n) = -1, \quad j = 3, 4, \ldots, n-1$.

1. Introduction and notation

All matrices considered here will be real.

A symmetric matrix is called elliptic if it has exactly one and simple positive eigenvalue. An elliptic matrix with all diagonal entries equal to zero is known

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PII: S 0 0 2 4 - 3 7 9 5 (9 9) 0 0 0 7 7 - 4

as a special elliptic matrix [1,2]. Let G_n be a simple and undirected graph with vertices $v_1, v_2, \ldots, v_n$. The adjacency matrix of a graph G_n is defined as a one-zero matrix $A(G) = (a_{ij})$, in which $a_{ij} = 1$ if and only if vertices v_i and v_j are adjacent.

Let $\det A(G_n)$ stand for the determinant of $A(G_n)$. The characteristic polynomial of G_n is the characteristic polynomial of its adjacency matrix, which is denoted by $P(G_n, \lambda)$. Since $A(G_n)$ is symmetric, its eigenvalues must be real. The kth largest eigenvalue $\lambda_k(G_n)$ of G_n is the kth largest root of $P(G_n, \lambda)$, where $1 \leqslant k \leqslant n$. Let G_n^c be the complement of G_n. Cao and Yuan showed that the second largest eigenvalue of a incomplete multpartite graph is greater than or equal to zero and there does not exist a graph such that the second largest eigenvalue of G_n lies in the interval $(-1, 0)$ (see [3]). In this note, we first give

$$\lambda_n(G_n) < -\frac{1 + \sqrt{1 + 4\frac{n-3}{n-1}}}{2}$$

$(n \geqslant 4)$ and then establish seven sufficient and necessary conditions such that $\lambda_2(G_n) = -1$, which yields that the second largest eigenvalue $\lambda_2(G_n)$ is either nonnegative or -1. This is the known result mentioned above. Next, we prove that $\lambda_3(G_n) = -1$ which implies $\lambda_j(G_n) = -1$, $j = 3, 4, \ldots, n-1$. Three conjectures are also posed here.

2. Lemma and results

Lemma 1 [1]. *Let $A = (a_{ij})$ be a special elliptic matrix. If A is nonsingular, then A has all off-diagonal entries different from zero.*

Lemma 2. *Let A be an $n \times n$ irreducible nonnegative matrix. Let $x = (x_1, x_2, \ldots, x_n)$ be a positive eigenvector and let $r = \max(x_i/x_j)$. Then*

$$s \leqslant \lambda_1(A) \leqslant S \tag{1}$$

and

$$(S/s)^{1/2} \leqslant r. \tag{2}$$

Moreover, equality holds in either (1) *or* (2) *if and only if $s = S$, where $S = \max s_i$, $s = \min s_i$, and s_i denotes the sum of elements of the ith row of A.*

Proof. The proof of Lemma 2 can be found in [6, p. 37]. □

Lemma 3. *Let G_n be a graph with at least four vertices. Then $\lambda_3(G) = -1$ iff G^c has at least four points and G^c is isomorphic to a complete bipartite graph plus some isolated vertices.*

Proof. This is the second part of Theorem 5 in [4]. □

Lemma 4. *Let G_n be a simply connected graph with $n \geqslant 3$ vertices but not complete. Then*

$$\lambda_n(G_n) \leqslant \lambda_n(K_{n-1}^1)$$

with the equality is true if and only if $G_n \cong K_{n-1}^1$, where K_{n-1}^1 is the graph obtained by the coalescence of a complete graph K_{n-1} of $n-1$ vertices with a path P_2 of length one at one of its vertices.

Proof. This is Theorem 2 of [5]. □

Lemma 5. *Let $K_{n-1}^1 (n \geqslant 4)$ be the graph defined in Lemma 4. Then*

$$n-2 < \lambda_1(K_{n-1}^1) < n-1,$$

$$\lambda_2(K_{n-1}^1) = \frac{n-3-\lambda_1(K_{n-1}^1) + \sqrt{(n-3-\lambda_1(K_{n-1}^1))^2 + 4(n-3)/\lambda_1(K_{n-1}^1)}}{2},$$

$$\lambda_j(K_{n-1}^1) = -1, \quad j = 3, \ldots, n-1,$$

$$\begin{aligned}\lambda_n(K_{n-1}^1) &= \frac{n-3-\lambda_1(K_{n-1}^1) - \sqrt{(n-3-\lambda_1(K_{n-1}^1))^2 + 4(n-3)/\lambda_1(K_{n-1}^1)}}{2} \\ &< \frac{-1-\sqrt{1+4(n-3)/(n-1)}}{2}.\end{aligned}$$

Proof. Since K_{n-1}^1 has a complete subgraph of $n-1$ vertices (say K_{n-1}), we have

$$\lambda_1(K_{n-1}) = n-2,$$
$$\lambda_j(K_{n-1}) = -1, \quad j = 2, \ldots, n-1.$$

Thus, by Cauchy's interlacing theorem [7], and by Lemma 2, it gives

$$n-2 < \lambda_1(K_{n-1}^1) < n-1, \quad \lambda_j(K_{n-1}^1) = -1, \quad j = 3, 4, \ldots, n-1. \tag{3}$$

Now we consider $\lambda_2(K_{n-1}^1)$ and $\lambda_n(K_{n-1}^1)$. Since the trace of $A(K_{n-1}^1)$ is zero, we have by (3)

$$\lambda_2(K_{n-1}^1) + \lambda_n(K_{n-1}^1) = n-3-\lambda_1(K_{n-1}^1). \tag{4}$$

On the other hand, it is easy to show that the determinant of $A(K^1_{n-1})$ is $(-1)^n(n-3)$, and therefore again by (3), it yields

$$\lambda_2(K^1_{n-1})\lambda_n(K^1_{n-1}) = -\frac{n-3}{\lambda_1(K^1_{n-1})}. \tag{5}$$

Combining (4) and (5), it leads to the following:

$$\lambda_2(K^1_{n-1}) = \frac{n-3-\lambda_1(K^1_{n-1}) + \sqrt{(n-3-\lambda_1(K^1_{n-1}))^2 + 4(n-3)/\lambda_1(K^1_{n-1})}}{2},$$

$$\begin{aligned}\lambda_n(K^1_{n-1}) &= \frac{n-3-\lambda_1(K^1_{n-1}) - \sqrt{(n-3-\lambda_1(K^1_{n-1}))^2 + 4(n-3)/\lambda_1(K^1_{n-1})}}{2}\\ &< \frac{-1-\sqrt{1+4(n-3)/(n-1)}}{2}.\end{aligned}$$

The proof is completed. □

Theorem 6. *Let G_n be a simply connected graph with $n \geqslant 4$ vertices but not complete. Then*

$$-\frac{n}{2} \leqslant \lambda_n(G_n) < -\frac{1+\sqrt{1+4(n-3)/(n-1)}}{2}.$$

Proof. The proof follows from Lemmas 4 and 5, and [8]. □

Remark 1. In [5], Yuan proved that

$$\lambda_n(K^1_{n-1}) \to -\frac{1+\sqrt{5}}{2}, \quad n \to \infty$$

But he did not establish the estimation with regard to n.

Theorem 7. *Let G_n be a simple graph. Then the following seven statements are equavilent*:

(a) $\lambda_2(G_n) = -1$
(b) G_n *is complete.*
(c) $\lambda_j(G_n) = -1, \quad j = 2, \ldots, n.$
(d) $\lambda_2(G_n) < 0.$
(e) $\lambda_n(G_n) = -1$, *and G_n is connected.*
(f) $A(G_n)$ *is a nonsingular special elliptic matrix.*
(g) $\lambda_1(G_n) = n-1.$

Proof. (a) $\Leftrightarrow$ (b) is by Theorem 1 of [3].

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are trivial.

(d) $\Rightarrow$ (e). Since $\lambda_1(G_n) > 0 > \lambda_2(G_n) \geqslant \lambda_j(G_n),\ j = 2, \ldots, n$, we know that $A(G_n)$ is a nonsingular special elliptic matrix. Theorefore by Lemma 1, G_n is complete. This implies that $\lambda_n(G_n) = -1$.

(e) $\Rightarrow$ (f). We need only to consider the case $n \geqslant 4$. By Theorem 6, $\lambda_n(G_n) = -1$ implies that G_n is complete. This yields $\lambda_1(G_n) = n - 1,\ \lambda_j(G_n) = -1$, $j = 2, \ldots, n$. i.e. $A(G_n)$ is a nonsingular special elliptic matrix.

(f) $\Rightarrow$ (g) is by Lemmas 1 and 2.

(g) $\Rightarrow$ (a) follows from Lemma 2. $\square$

Related to Theorem 7, we now pose two conjectures. They are true for $n \leqslant 8$. Both "only if" parts of them follow from Theorem 7.

Conjecture 1. Let G_n be a simple graph. Then G_n is complete if and only if

$$\det A(G_n) = (-1)^{n-1}(n-1).$$

Conjecture 2. Let G_n be a simple graph. Then G_n is complete if and only if

$$|\det A(G_n)| = \lambda_1(G_n) = n - 1.$$

Theorem 8. *Let G_n be a graph with at least four vertices. Then $\lambda_3(G_n) = -1$ implies that $\lambda_j(G_n) = -1,\ j = 3, \ldots, n-1$.*

Proof. By Lemma 3, it is easy to check that there exists a permutation matrix P such that

$$P^{\mathrm{T}} A(G_n) P = \begin{pmatrix} C_1 & 0 & J_{13} \\ 0 & C_2 & J_{23} \\ J_{13}^{\mathrm{T}} & J_{23}^{\mathrm{T}} & C_3 \end{pmatrix},$$

where C_i are the $k_i \times k_i$ matrices with all diagonal entries zero and all off-diagonal entries equal to one, $i = 1, 2, 3$, $\sum_{i=1}^{3} k_i = n, 0 \leqslant k_i \leqslant n$, and J_{ij} are the $k_i \times k_j, i, j = 1, 2, 3$, matrices with all entries equal to one.

We now consider the rank of matrix $P^{\mathrm{T}} A(G_n) P + I$. Let $r(Q)$ denote the rank of matrix Q. Then

$$
\begin{aligned}
r(P^{\mathrm{T}}A(G_n)P+I) &= r\begin{pmatrix} J_{11} & 0 & J_{13} \\ 0 & J_{22} & J_{23} \\ J_{13}^{\mathrm{T}} & J_{23}^{\mathrm{T}} & J_{33} \end{pmatrix} \\
&= r\begin{pmatrix} J_{11} & 0 & J_{13} \\ 0 & J_{22} & J_{23} \\ 0 & J_{23}^{\mathrm{T}} & 0 \end{pmatrix} \\
&= r\begin{pmatrix} J_{11} & 0 & J_{13} \\ 0 & J_{22} & J_{23} \\ 0 & 0 & -J_{33} \end{pmatrix} \\
&\leqslant 3,
\end{aligned}
$$

which states that $\lambda_3(G_n) = -1$ is a multiple root of matrix $A(G_n)$, whose multiplicity is at least $n-3$. For $\lambda_2(G_n)$ we have two cases:

(i) If $\lambda_2(G_n) = -1$, then by Theorem 7 it yields $\lambda_j(G_n) = -1,\ j = 2, 3, \ldots, n$.

(ii) $\lambda_2(G_n) \neq -1$ implies that $\lambda_2(G_n) \geqslant 0$ (by Theorem 7 again). Theorefore we have

$$\lambda_j(G_n) = -1, \quad j = 3, \ldots, n-1.$$

The proof is now completed. □

Conjecture 3. Let G_n be a graph with at least four vertices. Then $\lambda_k = -1$ $(2 \leqslant k \leqslant [n/2])$ implies that

$$\lambda_j(G_n) = -1, \quad j = k, k+1, \ldots, n-k+2.$$

For Conjecture 3, the cases $k = 2$, 3 are true (Theorems 7 and 8).

Example. Let

$$
A(G) = \begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}
$$

be the adjacency matrix of a graph with eight vertices. Then by MATHEMATICA we obtain

$$\begin{aligned}
\lambda_1(G) &= 5.24384,\\
\lambda_2(G) &= 1.60317,\\
\lambda_3(G) &= -0.182062,\\
\lambda_4(G) &= -0.9999,\\
\lambda_5(G) &= -1,\\
\lambda_6(G) &= -1,\\
\lambda_7(G) &= -1.53035,\\
\lambda_8(G) &= -2.1346.
\end{aligned}$$

On the other hand, since it is easy to verify that $r(I + A(G)) = 5$ (the rank of matrix $I + A(G)$), we infer that

$$\lambda_4(G) = \lambda_5(G) = \lambda_6(G) = -1.$$

Compared with Conjecture 3, this is the special case that $k = 4,\ n = 8$.

Remark 2. Let G_n be the graph defined in Theorem 8 and have l edges. If $\lambda_3 = -1$, then $\lambda_1, \lambda_2, \lambda_n$ are given by:

$$\begin{aligned}
&\lambda_1 + \lambda_2 + \lambda_n = n - 3,\\
&\lambda_1^2 + \lambda_2^2 + \lambda_n^2 = 2l - (n - 3),\\
&\lambda_1\lambda_2\lambda_n = \begin{vmatrix} C_1 & 0 & J_1 \\ 0 & C_2 & J_2 \\ J_1^{\mathrm{T}} & J_2^{\mathrm{T}} & C_3 \end{vmatrix} = (-1)^n \det A(G_n).
\end{aligned}$$

It is not difficult to obtain the expression of $\det A(G_n)$.

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