
Volume 23, Number 3, March 2010
ISSN 0893-9659
ELSEVIER
Applied Mathematics Letters
an international journal of rapid publication
Editorial Board
Ravi P. Agarwal
R. Ahlswede
Wolfgang Alt
A. Ambrosetti
H. T. Banks
Claude Bardos
A. Bensoussan
Paul Boggs
F. Brezzi
C. Cergignani
Nick Cercone
Bruno Codenotti
Thomas F. Coleman
Jerome R. Cox Jr
R. Dautray
A. W. M. Dress
I. Ekeland
B. Fuchssteiner
James Glimm
K. Glover
Roland Glowinski
I. Gohberg
R. L. Graham
G. J. Habetler
M. Iri
Rudolf E. Kalman
A. Kurzhanski
T. J. Laffey
Leslie Lamport
Peter D. Lax
G. Lebon
Mark Levi
Pierre Louis Lions
William F. Lucas
P. Markowich
Bernard J. Matkowsky
Gordon McCalla
A. Nachman
J.R. Ockendon
J. Tinsley Oden
Steven A. Orszag
Eduardo L. Ortiz
L. Pandolfi
Jong-Shi Pang
N. Papamichael
Ervin Y. Rodin
Donald J. Rose
Thomas L. Saaty
Jean Claude Saut
Martin H. Schultz
Denis Serre
R. Temam
Samuel C. C. Ting
Joseph F. Traub
Alan Tucker
Paul Michael Van Dooren
Graeme C. Wake
Tarynn M. Witten
Lotfi A. Zadeh
Available online at
ScienceDirect
www.sciencedirect.com
NOW included in your subscription:
ELECTRONIC ACCESS
www.elsevier.com/locate/elecacc

Applied Mathematics Letters 23 (2010) 291–298

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Counting the number of spanning trees in a class of double fixed-step loop networks[☆]

Talip Atajan[a,∗], Naohisa Otsuka[b], Xuerong Yong[c]

[a] School of Science and Engineering, Tokyo Denki University, Hatoyama-machi, Hiki-gun, Saitama 350-0394, Japan
[b] Division of Science, Tokyo Denki University, Hatoyama-machi, Hiki-gun, Saitama 350-0394, Japan
[c] Department of Mathematical Sciences, University of Puerto Rico, Mayaguez, PR 00681, USA

A R T I C L E I N F O

Article history:
Received 22 April 2009
Accepted 22 April 2009

A B S T R A C T

A double fixed-step loop network, $\vec{C}_n^{p,q}$, is a digraph on n vertices $0, 1, 2, \ldots, n-1$ and for each vertex i $(0 < i \le n-1)$, there are exactly two arcs going from vertex i to vertices $i+p$, $i+q \pmod n$. Let $p < q < n$ be positive integers such that $(q-p) \nmid n$ and $(q-p)|(k_0 n - p)$ or $(q-p)|n$ (where $k_0 = min\{k|(q-p)|(kn-p), k = 1, 2, 3, \ldots\}$ and $gcd(q, p) = 1$. In this work we derive a formula for the number of spanning trees, $T(\vec{C}_n^{p,q})$, with constant or nonconstant jumps and prove that $T(\vec{C}_n^{p,q})$ can be represented asymptotically by the mth-order 'Fibonacci' numbers. Some special cases give rise to the formulas obtained recently in [Z. Lonc, K. Parol, J.M. Wojciechowski, On the number of spanning trees in directed circulant graphs, Networks 37 (2001) 129–133; X. Yong, F.J. Zhang, An asymptotic behavior of the complexity of double fixed step loop networks, Applied Mathematics. A Journal of Chinese Universities. Ser. B 12 (1997) 233– 236; X. Yong, Y. Zhang, M. Golin, The number of spanning trees in a class of double fixed-step loop networks, Networks 52 (2) (2008) 69–87].

1. Introduction

A directed circulant graph, $\vec{C}^{s_1,s_2,\ldots,s_k}$, is a digraph on n vertices $0, 1, 2, \ldots, n-1$ and for each vertex i $(0 < i \le n-1)$, there are k arcs from vertex i to vertices $i+s_1, i+s_2, \ldots, i+s_k \pmod n$. A *double fixed-step loop network*, $\vec{C}_n^{p,q}$, is the directed circulant graph where each vertex has exactly two arcs leaving from it. Since this class of networks arises in the fields of design and analysis of local area networks, multi-module memory organizations and supercomputer architectures etc., there has been active research on their parameters such as the number of spanning trees, diameter, average distance, etc. [1–5] (in applications these parameters are closely related to the bandwidth of a given network). And, among the parameters, the number of spanning trees is essential and also characterizes the reliability of a network in the presence of line faults [6]. Finding the exact value or the asymptotic number of spanning trees of a graph is usually not easy and, therefore, research into this parameter has been focusing on special classes of graphs and different techniques are being developed for different classes of graphs [7,2,8].

Before starting, we should point out that, theoretically, Kirchhoff's *Matrix Tree Theorem* [9] can be modified to calculate the number of spanning trees in any digraph G through evaluating the determinant of any $(n-1)$th-order sub-matrix of its 'Laplace' matrix. However, counting the number of spanning trees by calculating the determinant is *infeasible* for large graphs. Because of this, in the last few decades, researchers have developed techniques for getting around the difficulty

☆ The research of the first two authors was supported partially by the Tokyo Denki University. The third author's work was supported in part by DIMACS and the University of Puerto Rico at Mayaguez.

∗ Corresponding author.
E-mail addresses: talip@j.dendai.ac.jp (T. Atajan), otsuka@j.dendai.ac.jp (N. Otsuka), xryong@math.uprm.edu (X. Yong).

0893-9659/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2009.04.006

and paid great attention to deriving explicit or possibly recursive formulas [10–13]. For general circulant (di-)graphs, some recent formulas are seen in [1,4,5]. But, unfortunately, if the biggest jump s_k is large, it is not easy to derive the recurrence relations to get the number of spanning trees in a given directed circulant because the order of the recurrence relation increases exponentially with s_k. This motivated us to come up with the present work.

In Section 2 we derive some basic results which will be used to obtain our main results. Section 3 focuses on finding the formulas for the number of spanning trees in $\vec{C}_n^{p,q}$, with constant or nonconstant jumps. For positive integers p, q, n with $p < q < n$, some formulas for the numbers are also derived. Section 4 involves finding an approximation formula for $T(\vec{C}_n^{p,q})$.

We should address that there is a formula

$$T\left(\vec{C}_n^{p,q}\right) = \prod_{j=1}^{n-1}(2 - \varepsilon^{pj} - \varepsilon^{qj})$$

where $\varepsilon = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$, $\mathrm{i} = \sqrt{-1}$. But, unfortunately, direct calculation of this formula is not efficient in applications because of the sin and cos in ε^j. And the computation is ill-conditioned. The aim of this work is therefore transforming the formula into something more convenient and interesting.

In this work, for convenience, we use $x \dagger y$ to denote that x cannot be divided by y and C_n^k to denote the binomial coefficients $\binom{n}{k}$.

2. Basic lemmas

We start this section by considering the divisible properties of the positive integers involved to derive our preliminaries. The idea is to proceed by applying some basic facts from combinatorics. We need to introduce the matrix B below whose properties will play essential roles in establishing the formulas for the numbers of spanning trees in the double fixed-step loop networks, $\vec{C}_n^{p,q}$.

Definition 1. For any positive integers $p < q < n$, let $B = (b_{i,j})$ be the $(n-1) \times (+\infty)$ matrix satisfying that $b_{i,j} = pi + (q-p)j$, $i = 1, 2, \ldots, n-1$; $j = 0, 1, 2, \ldots$.

Note that B is a positive integer matrix and with this definition we have the following Lemma 1.

Lemma 1. *For any positive integers $p < q < n$,*

(a) $n | b_{i,i\frac{k_0 n - p}{q-p}}$, *if* $(q-p) \dagger n$ *and* $(q-p)|(k_0 n - p)$ $i = 1, 2, \ldots,$

(b) $n | b_{i,j}$ *if* $(q-p)|n$, $gcd(q-p, p) = 1$, $i = k(q-p)$, $j = k(n-p)$, $k = 1, 2, \ldots,$

(c) $n \dagger b_{i,j}$ *if* $(q-p)|n$, $gcd(q-p, p) = 1$, $i \neq k(q-p)$, $j = 0, 1, 2, \ldots, k = 1, 2, \ldots,$

where $k_0 = \min\{k | (q-p)|(kn-p), k = 1, 2, 3, \ldots\}$.

Proof. (a) By the definition of the elements $b_{i,j}$ of B, we have immediately that

$$\begin{aligned} b_{i,i\frac{k_0 n - p}{q-p}} &= ip + (q-p)\left[i\frac{k_0 n - p}{q-p}\right] \\ &= ip + ik_0 n - ip \\ &= ik_0 n, \end{aligned}$$

which proves (a). To show (b), repeatedly applying the same strategy as was used for proving (a) yields

$$\begin{aligned} b_{i,j} &= ip + (q-p)j = k(q-p)p + (q-p)k(n-p) \\ &= k(q-p)p + (q-p)kn - (q-p)kp \\ &= (q-p)kn. \end{aligned}$$

This implies that n is a divisor of $b_{i,j}$ when $i = k(q-p)$, $j = k(n-p)$, $k = 1, 2, \ldots$ and therefore (b) is proven.

We now prove (c) by contradiction. If $b_{i,j}$ could be divided by n, then there would exist integers m_1, m_2 such that $\frac{b_{i,j}}{n} = m_1$, $\frac{n}{q-p} = m_2$ and $b_{i,j} = ip + (q-p)j = m_1 n$, equivalent to

$$\frac{ip}{q-p} + j = m_1 \frac{n}{q-p} = m_1 m_2.$$

This is a contradiction because the right side is a fraction but the left side is an integer. Putting all the above together the proof of the lemma is completed. □

If $gcd(q-p, p) = d > 1$ and $(q-p)|n$ then there exist integers k_1, k_2 and k_3 such that $q = dk_1$, $p = dk_2$ and $n = (q-p)k_3 = d(k_1 - k_2)k_3$. So $gcd(n, q, p) = d > 1$ implies that the graph is disconnected and therefore $T(\vec{C}_n^{p,q}) = 0$. In the future we will assume that $gcd(p, q) = 1$ and n is not divisible by $q - p$.

We will consider the elements $b_{i,j}$ that are divisible by n.

Lemma 2. *For any positive integers $p < q < n$, if the elements $b_{i,j}$ of B satisfy that $n|b_{i,j}$ then $n|b_{i,j\pm k\frac{n}{d}}$ for $k = 0, 1, 2, \ldots$, where $d = gcd(n, q-p)$.*

Proof. This lemma can be proven in the same way as the one used in the proof of Lemma 1.

$$\begin{aligned} b_{i,j\pm k\frac{n}{d}} &= ip + (q-p)\left(j \pm k\frac{n}{d}\right) \\ &= ip + (q-p)j \pm (q-p)k\frac{n}{d} \\ &= b_{i,j} \pm \frac{q-p}{d}kn. \quad \square \end{aligned}$$

Lemma 3. *For any positive integers $p < q < n$,*

(a) $b_{i,i\frac{k_0 n-p}{q-p} \bmod \frac{n}{d}}$, *if $(q-p) \dagger n$ and $(q-p)|(k_0 n - p)$, $i = 1, 2, 3, \ldots$*

(b) $b_{k(q-p),k(n-p) \bmod \frac{n}{q-p}}$, *if $(q-p)|n$ and $gcd(q-p, p) = 1$, $k = 1, 2, 3, \ldots$*

is the first element in the ith ($k(q-p)$ th) row that can be divided by n, where $d = gcd(n, q-p)$, $k_0 = \min\{k|(q-p)|(kn-p), k = 1, 2, 3, \ldots\}$.

Proof. To prove (a), without loss of generality, we may consider the mth row of B, where $(q-p) \dagger n$ and $(q-p)|(k_0 n - p)$ then, from Lemma 1 $n|b_{m,m\frac{k_0 n-p}{q-p}}$ and if $\frac{n}{d} < m\frac{k_0 n-p}{q-p}$ then from Lemma 2 we have that $n|b_{m,m\frac{k_0 n-p}{q-p}-\frac{n}{d}}$ and if we still have $\frac{n}{d} < m\frac{k_0 n-p}{q-p} - \frac{n}{d}$ then from Lemma 2 again we have $n|b_{m,m\frac{k_0 n-p}{q-p}-\frac{n}{d}-\frac{n}{d}}$. We can obtain the first divisible element by repeating the same process until

$$0 \le m\frac{k_0 n - p}{q-p} - \frac{n}{d} - \frac{n}{d} - \cdots \le \frac{n}{d}.$$

This is equivalent to $m\frac{k_0 n-p}{q-p} \bmod \frac{n}{d}$, implying that $b_{m,m\frac{k_0 n-p}{q-p} \bmod \frac{n}{d}}$ is the first element in the mth row that can be divided by n. (b) can be proven in the same way. $\square$

3. The number of spanning trees in the networks

In this section, making use of the basics obtained in the previous section, we develop a method for calculating the number of spanning trees in $\vec{C}_n^{p,q}$ with constant or nonconstant jumps. We start by recalling the following known lemma.

Lemma 4 ([4]). *Let n, p and q be any positive integers and*

$$f(x) = \prod_{j=0}^{n-1}(x - \varepsilon^{pj} - \varepsilon^{qj}) = \delta_0 x^n + \delta_1 x^{n-1} + \delta_2 x^{n-2} + \cdots + \delta_{n-1}x + \delta_n, \quad \delta_0 = 1. \tag{1}$$

Then

$$T(\vec{C}_n^{p,q}) = f'(2) = \sum_{j=0}^{n-1}(n-j)\delta_j 2^{n-j-1}, \tag{2}$$

where $\varepsilon = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$.

For some special $p < q < n$, the following Lemma 5 gives us the binomial expression formula of the power sum, $S_m = \sum_{j=1}^{n}(\varepsilon^{pj} + \varepsilon^{qj})^m$, of polynomial (1).

Lemma 5. *For any positive integers $p < q < n$,*

$$S_m = \begin{cases} n\displaystyle\sum_{i=0}^{d-1} C_m^{(m\frac{k_0 n-p}{q-p}) \bmod \frac{n}{d} + \frac{n}{d}i}, & \text{if } (q-p) \dagger n, \text{ and } (q-p)|(k_0 n - p), \\ n\displaystyle\sum_{i=0}^{q-p-1} C_m^{(v(n-p)) \bmod \frac{n}{q-p} + \frac{n}{q-p}i}, & \text{if } (q-p)|n \text{ and } gcd(q-p, p) = 1, m = v(q-p), \\ 0, & \text{if } (q-p)|n \text{ and } gcd(q-p, p) = 1, m \ne v(q-p), \end{cases}$$

where $d = gcd(n, q-p)$, $k_0 = \min\{k|(q-p)|(kn-p), k = 1, 2, 3, \ldots\}$ and $v = 1, 2, 3, \ldots$.

Proof.

$$\begin{aligned}
S_m &= \sum_{j=1}^{n} (\varepsilon^{pj} + \varepsilon^{qj})^m \\
&= \sum_{j=1}^{n} (C_m^0 \varepsilon^{jpm} + C_m^1 \varepsilon^{j(p(m-1)+q)} + C_m^2 \varepsilon^{j(p(m-2)+2q)} + \cdots + C_m^{m-1} \varepsilon^{j(p+(m-1)q)} + C_m^m \varepsilon^{j(mq)}) \\
&= \sum_{j=1}^{n} (C_m^0 \varepsilon^{j(mp+0(q-p))} + C_m^1 \varepsilon^{j(mp+(q-p))} + C_m^2 \varepsilon^{j(mp+(q-p)2)} + \cdots + C_m^m \varepsilon^{j(mp+m(q-p))}) \\
&= \sum_{j=1}^{n} (C_m^0 \varepsilon^{jb_{m,0}} + C_m^1 \varepsilon^{jb_{m,1}} + C_m^2 \varepsilon^{jb_{m,2}} + \cdots + C_m^m \varepsilon^{jb_{m,m}}) \\
&= C_m^0 \sum_{j=1}^{n} \varepsilon^{jb_{m,0}} + C_m^1 \sum_{j=1}^{n} \varepsilon^{jb_{m,1}} + C_m^2 \sum_{j=1}^{n} \varepsilon^{jb_{m,2}} + \cdots + C_m^m \sum_{j=1}^{n} \varepsilon^{jb_{m,m}},
\end{aligned}$$

where $b_{m,k},\ k = 0, 1, 2, \ldots,$ are the elements of B. Noting that

$$\sum_{j=1}^{n} \varepsilon^{jb_{m,k}} = \begin{cases} n, & \text{if } n | b_{m,k}, \\ 0, & \text{otherwise}, \end{cases}$$

we see that the claim of the lemma is true from the above discussions. □

It is well known that if $\alpha_i,\ i = 1, 2, \ldots, n$, are the roots of a polynomial

$$p(x) = \beta_0 x^n + \beta_1 x^{n-1} + \beta_2 x^{n-2} + \cdots + \beta_{n-1} x + \beta_n, \quad \beta_0 = 1,$$

then $S_k = \beta_1^k + \beta_2^k + \cdots + \beta_n^k$ is called the kth power sum of $p(x)$. The power sums S_k, $k = 1, 2, \ldots, n$, can be calculated, so the coefficients β_k, $k = 1, 2, \ldots, n$, can be derived using *Newton's Identities*

$$S_k\beta_0 + S_{k-1}\beta_1 + S_{k-2}\beta_2 + \cdots + +S_1\beta_{k-1} + k\beta_k = 0, \tag{3}$$

In Lemma 5, we have obtained explicit expressions for the power sums of polynomial (1). The following Theorem 6 can be obtained by combining Lemma 5 with *Newton's Identities*. For some special integers $p < q < n$, we can calculate the number of spanning trees by using the following Theorem 6.

Theorem 6. *For any positive integers $p < q \le n - 1$ if $(q - p) \dagger n$, $(q - p) | (k_0 n - p)$ then*

$$T(\vec{C}_n^{p,q}) = \sum_{i=0}^{n-1} (n - i)\delta_i 2^{n-1-i}, \tag{4}$$

where

$$\begin{aligned}
&\delta_k = \frac{1}{k}(S_k - S_{k-1}\delta_1 - S_{k-2}\delta_2 - \cdots - S_1\delta_{k-1}), \\
&S_k = \sum_{i=0}^{d-1} C_k^{(k\frac{n-p}{q-p}) \bmod \frac{n}{d} + \frac{n}{d} i}, \quad k = 1, 2, \ldots, n-1 \\
&d = gcd(n, q - p).
\end{aligned}$$

We can see from the expressions in Lemma 5 and Theorem 6 that if $(q - p) | n$ then the formula for the number of spanning trees can be calculated through the following Corollary 1.

Corollary 1. *For any positive integers $p < q < n$ if $(q - p) | n$ and $div(q - p, p) = 1$ then*

$$T(\vec{C}_n^{p,q}) = \sum_{i=0}^{n-1} (n - i)\delta_i 2^{n-1-i}, \tag{5}$$

where

$$\begin{aligned}
&\delta_k = \frac{1}{k}(S_k - S_{k-1}\delta_1 - S_{k-2}\delta_2 - \cdots - S_1\delta_{k-1}), \\
&S_m = \begin{cases} n \displaystyle\sum_{i=0}^{d-1} C_m^{(k(\frac{n}{q-p} - p)) \bmod \frac{n}{d} + \frac{n}{d} i} & \text{if } m = k(q - p),\ k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

We will illustrate our techniques, Theorem 6 and Corollary 1, by evaluating the following two examples.

Example 1. Case $T(\vec{C}_{20}^{3,10})$.

Here $p = 3, q = 10, n = 20, d = gcd(n, q - p) = 1$ and $k_0 = \min\{k|\ (q-p)|(kn-p), k = 1, 2, 3, \ldots\} = 4$; then from Theorem 6

$$T(\vec{C}_{20}^{3,10}) = \sum_{i=1}^{19}(20-i)\delta_i 2^{19-i}, \tag{6}$$

where

$$\delta_k = \frac{1}{k}(S_k - S_{k-1}\delta_1 - S_{k-2}\delta_2 - \cdots - S_1\delta_{k-1}), \tag{7}$$

$$\begin{aligned} S_k &= 20C_k^{(k\frac{4\times 20-3}{10-3})\text{mod } \frac{20}{1}} \\ &= 20C_k^{(11k)\text{mod } 20}, \quad k = 1, 2, \ldots, 19. \end{aligned} \tag{8}$$

For $k = 1, 2, \ldots, 19$, we have the following results from calculation:

$$\begin{aligned} &S_1 = 20C_1^{11} = 0, \quad S_2 = 20C_2^2 = 20, \quad S_3 = 20C_3^{13} = 0, \quad S_4 = 20C_4^4 = 20, \\ &S_5 = 20C_5^{15} = 0, \quad S_6 = 20C_6^6 = 20, \quad S_7 = 20C_7^{17} = 0, \quad S_8 = 20C_8^8 = 20, \\ &S_9 = 20C_9^{19} = 0, \quad S_{10} = 20C_{10}^{10} = 20, \quad S_{11} = 20C_{11}^1 = 220, \quad S_{12} = 20C_{12}^{12} = 20, \\ &S_{13} = 20C_{13}^3 = 5720, \quad S_{14} = 20C_{14}^{14} = 20, \quad S_{15} = 20C_{15}^5 = 60\,060, \quad S_{16} = 20C_{16}^{16} = 20, \\ &S_{17} = 20C_{17}^7 = 1166\,880, \quad S_{18} = 20C_{18}^{18} = 20, \quad S_{19} = 20C_{19}^9 = 923\,780. \end{aligned}$$

Then these results plug into (7), that is

$$\delta_1 = -\frac{1}{1}S_1 = 0, \quad \delta_2 = -\frac{1}{2}(S_2 + \delta_1 S_1) = -10, \quad \delta_3 = -\frac{1}{3}(S_3 + \delta_1 S_2 + \delta_2 S_1) = 0,$$

and repeating this process we have

$$\begin{aligned} &\delta_4 = 45, \quad \delta_5 = 0, \quad \delta_6 = -120, \quad \delta_7 = 0, \quad \delta_4 = 45, \quad \delta_5 = 0, \\ &\delta_6 = -120, \quad \delta_7 = 0, \quad \delta_8 = 210, \quad \delta_9 = 0, \quad \delta_{10} = -252, \quad \delta_{11} = -20, \\ &\delta_{12} = 210, \quad \delta_{13} = -240, \quad \delta_{14} = -120, \quad \delta_{15} = -504, \quad \delta_{16} = 45, \quad \delta_{17} = -46\,000, \\ &\delta_{18} = -10, \quad \delta_{19} = 506\,200. \end{aligned}$$

And from (6)

$$T(\vec{C}_{20}^{3,10}) = 10\,485\,760 = 20 \times 524\,288.$$

Example 2. Case $T(\vec{C}_{32}^{13,21})$.

Here $p = 13, q = 21, n = 32$, and $d = (n, q - p) = 8$; then from Corollary 1, we have that

$$S_8 = 32\sum_{i=0}^{7} C_m^{(1(32-13))\text{mod}4+4i} = 32(C_8^3 + C_8^7) = 2048,$$

$$S_{16} = 32\sum_{i=0}^{7} C_{16}^{(2(32-13))\text{mod}4+4i} = 32(C_{16}^2 + C_{16}^6 + \cdots + C_{16}^{14}) = 520\,192,$$

$$S_{24} = 32\sum_{i=0}^{7} C_{24}^{(3(32-13))\text{mod}4+4i} = 32(C_{24}^1 + C_{24}^5 + \cdots + C_{24}^{21}) = 134\,217\,728.$$

And the nonzero coefficents, δ_i, are

$$\delta_8 = -256, \quad \delta_{16} = 256, \quad \delta_{24} = -65\,536.$$

Then from (5), we have

$$T(\vec{C}_{32}^{13,21}) = 68\,719\,476\,736 = 32 \times 2147\,483\,648.$$

4. The formula and the mth-order 'Fibonacci' recurrence relation

In this section we fix p, q and show that, when n, m are large and $gcd(n, p) = 1$ (or $gcd(n, q) = 1$), the number of spanning trees $T(\vec{C}_n^{p,q})$ can be approached asymptotically by the mth-order 'Fibonacci' number [14]. Our derivation also implies that the two quantities, the number of spanning trees $T(\vec{C}_n^{p,q})$ and the mth-order 'Fibonacci' number, share the same asymptotic growth rate. The following lemma is crucial for deriving the formulas.

Lemma 7. *For any given positive integer m, the polynomial*

$$z^m - z^{m-1} - \cdots - z - 1 = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_m),$$

has m distinct roots and has exactly one real positive root (which is the largest in modulus of all the roots), $\alpha_1(m)$, with $1 < \alpha(m) < 2$. Furthermore, when m tends to $+\infty$, $\alpha_1(m)$ is monotonically increasing and approaches 2.

Proof. The first part of the claim is due to Knuth. See, for example, [14] (page 161). The validity for the second part can be easily proven by making use of the Perron–Frobenius Theorem in nonnegative matrix theory because the polynomial can be viewed as the characteristic polynomial of an $m \times m$ irreducible nonnegative matrix. (A proof is seen in [15].) $\square$

Since we assume that $gcd(n, p) = 1$, and that $1 \le p < q < n$, there exist (the smallest in modulus) two integers u, v such that $un + vp = 1$. Let $r = vq \bmod n$ and $\varepsilon_1 = \varepsilon^p$. Then $\varepsilon = \varepsilon_1^v$ and $\{\varepsilon_1, \varepsilon_1^2, \ldots, \varepsilon_1^{n-1}\}$ is a permutation of $\{\varepsilon, \varepsilon^2, \ldots, \varepsilon^{n-1}\}$. Noticing that $\prod_{j=1}^{n-1}(1 - \varepsilon_1^j) = n$ and that $|\prod_{j=1}^{n-1} \varepsilon_1^{(n-r)j}| = 1$ we have

$$\begin{aligned}
T(\vec{C}_n^{p,q}) &= \prod_{j=1}^{n-1}(2 - \varepsilon^{pj} - \varepsilon^{qj}) = \prod_{j=1}^{n-1}(2 - \varepsilon_1^j - \varepsilon_1^{vqj}) \\
&= \left|\prod_{j=1}^{n-1}(2 - \varepsilon_1^j - \varepsilon_1^{rj}) \prod_{j=1}^{n-1}(\varepsilon_1^{(n-r)j})\right| \\
&= \left|\prod_{j=1}^{n-1}(2\varepsilon_1^{(n-r)j} - \varepsilon_1^{(n-r+1)j} - 1)\right| \\
&= \left|\prod_{j=1}^{n-1}(\varepsilon_1^{mj} - \varepsilon_1^{(m-1)j} - \cdots - \varepsilon_1^j - 1) \prod_{j=1}^{n-1}(1 - \varepsilon_1^j)\right| \\
&= n\left|\prod_{j=1}^{n-1}(\varepsilon_1^{mj} - \varepsilon_1^{(m-1)j} - \cdots - \varepsilon_1^j - 1)\right|
\end{aligned}$$

where $m = n - r$. From Lemma 7, we have that $\varepsilon_1^{mj} - \varepsilon_1^{(m-1)j} - \cdots - \varepsilon_1^j - 1 = \prod_{i=1}^m(\varepsilon_1^j - \alpha_i)$. So applying the identity $\prod_{j=1}^{n-1}(\varepsilon_1^j - x) = (-1)^{n-1}\sum_{j=0}^{n-1} x^j$ yields

$$\begin{aligned}
T(\vec{C}_n^{p,q}) &= n\left|\prod_{j=1}^{n-1}(\varepsilon_1^{mj} - \varepsilon_1^{(m-1)j} - \cdots - \varepsilon_1^j - 1)\right| \\
&= n\left|\prod_{j=1}^{n-1}\prod_{i=1}^{m}(\varepsilon_1^j - \alpha_i)\right| \\
&= n\left|\prod_{i=1}^{m}\sum_{j=0}^{n-1}\alpha_i^j\right| \\
&= n\prod_{i=1}^{m}\frac{1 - \alpha_i^n}{1 - \alpha_i}.
\end{aligned}$$

In [16] it is shown that $\lim_{n\to+\infty}\left(T(\vec{C}_n^{p,q})\right)^{1/n} = 2$. This implies $\left(\prod_{i=1}^m(\alpha_i^n - 1)\right)^{1/n} \to 2$. So by Lemma 7, when n is large, we achieve

$$\begin{aligned}
\prod_{i=1}^{m}(1 - \alpha_i^n) &= 1 - \sum_{j=1}^{m}\alpha_j^n + \sum_{j_1<j_2}\left(\alpha_{j_1}\alpha_{j_2}\right)^n + \cdots + (-1)^{m-1}\sum_{j_1<j_2<\cdots<j_m}\left(\alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_m}\right)^n \\
&\sim -\sum_{j=1}^{m}\alpha_j^n.
\end{aligned}$$

Therefore, since $\prod_{i=1}^{m}(1-\alpha_i) = -(m-1) = -(n-r-1)$, this yields

$$T(\vec{C}_n^{p,q}) = n\prod_{i=1}^{m}\frac{1-\alpha_i^n}{1-\alpha_i} \sim \frac{n}{n-r-1}\sum_{j=1}^{m}\alpha_j^n \sim F_n.$$

On the other hand, if we let $F_t = \sum_{j=1}^{m}\alpha_j^t$, then $\alpha_i^m - \alpha_i^{m-1} - \cdots - \alpha_i - 1 = 0$ implies for all $t > 0$ that

$$F_{t+m} = F_{t+m-1} + F_{t+m-2} + \cdots + F_t.$$

This is the mth-order 'Fibonacci' recurrence relation. Note that when t is large, $F_t \longrightarrow \alpha_1^t$ and that *when n tends to $+\infty$, m is forced to tend to $+\infty$*. Summarizing the above discussions leads to the following theorem.

Theorem 8. *Let $\gcd(n,p) = 1$ and u, v be the smallest (in modulus) integers such that $un + vp = 1$, and let $m = n - r$ where $r = vq \bmod n$. Then the number of spanning trees, $T(\vec{C}_n^{p,q})$, and the mth-order 'Fibonacci' sequence, F_n, have the same asymptotic growth rate. To be precise, we have the asymptotic formula $T(\vec{C}_n^{p,q}) \sim F_n$ and, if $\gcd(n+1,p) = 1$,*

$$\lim_{n\to+\infty}\frac{T(\vec{C}_{n+1}^{p,q})}{T(\vec{C}_n^{p,q})} = \lim_{n\to+\infty}\frac{F_{n+1}}{F_n} = \lim_{m\to+\infty}\alpha_1(m) = 2$$

where $F_{t+m} = F_{t+m-1} + F_{t+m-2} + \cdots + F_t$ and $F_t = \sum_{j=1}^{m}\alpha_j^t$ (it can be shown that the initial numbers are $F_0 = m$, $F_j = 2^j - 1$, $1 \le j \le m-1$).

Remark 1. When $p = 1$ Theorem 8 generates the main result obtained in [15], that is,

$$\lim_{n\to+\infty}\frac{T(\vec{C}_{n+1}^{1,q})}{T(\vec{C}_n^{1,q})} = 2.$$

Remark 2. It is shown in [8] that $T(\vec{C}_n^{p,q}) \le T(\vec{C}_n^{1,2})$. However, Theorem 8 implies that they have the same average growth rate, i.e., $\lim_{n\to+\infty}\frac{T(\vec{C}_{n+1}^{p,q})}{T(\vec{C}_n^{p,q})} = \lim_{n\to+\infty}\frac{T(\vec{C}_{n+1}^{1,2})}{T(\vec{C}_n^{1,2})} = 2$.

Example 3. When $p = 1, q = 2$, applying the identity $\prod_{j=1}^{n-1}(x - \varepsilon^j) = \sum_{j=0}^{n-1}x^j$ and by the formula we have that

$$\begin{aligned} T(C_n^{1,2}) &= \prod_{j=1}^{n-1}(2-\varepsilon^j-\varepsilon^{2j}) = \prod_{j=1}^{n-1}(2+\varepsilon^j)(1-\varepsilon^j) \\ &= (-1)^{n-1}\prod_{j=1}^{n-1}(-2-\varepsilon^j)\prod_{j=1}^{n-1}(1-\varepsilon^j) \\ &= n\left(\frac{2^n+(-1)^n}{3}\right), \end{aligned}$$

seeing again that $\lim_{n\to+\infty}\frac{T(\vec{C}_{n+1}^{1,2})}{T(\vec{C}_n^{1,2})} = 2$.

We would like to point out that the authors of [8] define $T(\vec{C}_n^{p,q})$ to be $\frac{1}{n}\prod_{j=1}^{n-1}(2-\varepsilon^{pj}-\varepsilon^{qj})$ and prove that it is equal to $\left(\frac{2^n+(-1)^n}{3}\right)$. We see that Example 3 gives an alternative, but much more simpler, proof of the formula.

5. Conclusion and open question

In this work, for any positive integers p, q, n with $p < q < n$, we derived an explicit formula for counting the number of spanning trees in a class of double fixed-step loop networks, $\vec{C}_n^{q,p}$, with constant or nonconstant jumps. We then proved that the number of spanning trees can be approached by an approximation formula which is based on the mth-order 'Fibonacci' numbers. Some special cases generate the formulas obtained in previous papers.

One interesting aim would be to simplify the formulas in Theorem 6.

References

[1] T. Atajan, X. Yong, H. Inaba, Further analysis of the spanning tree formulas in circulant graphs, Discrete Mathematics 306 (2006) 2817–2827.
[2] P. Erdös, D.F. Hsu, Distributed loop networks with minimum transmission delay, Theoretical Computer Science 100 (1992) 223–241.
[3] M.T. Liu, Distributed loop computer networks, Advanced in Computers 17 (1978) 163–221.
[4] X. Yong, Y. Zhang, M. Golin, The number of spanning trees in a class of double fixed-step loop networks, Networks 52 (2) (2008) 69–87.
[5] Y. Zhang, X. Yong, M. Golin, The number of spanning trees in circulant graphs, Discrete Mathematics 223 (2000) 337–350.
[6] F.K. Hwang, A survey on double loop networks, in: Proceedings DIMACS Workshop on Reliability of Computer and Communication Networks, 1989.
[7] T. Atajan, N. Otsuka, X. Yong, The spanning trees formulas in a class of double fixed-step loop networks, in: The Proceedings of SIAM Workshop on Analytic Algorithmics and Combinatorics, ANALCO09, Jan. 2009, New York City, USA.
[8] Z. Lonc, K. Parol, J.M. Wojciechowski, On the number of spanning trees in directed circulant graphs, Networks 37 (2001) 129–133.
[9] G. Kirchhoff, Über die Auflösung der gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer Ströme geführt wird, Ann. Phys. Chem. 72 (1847) 497–508.
[10] G. Baron, H. Prodinger, R. Tichy, F. Boesch, J. Wang, The number of spanning trees in the square of a cycle, The Fibonacci Quarterly 23 (1985) 258–264.
[11] Xiebin Chen, The number of spanning trees in directed circulant graphs with non-fixed jumps, Discrete Mathematics 307 (2007) 1873–1880.
[12] R. Vohra, L. Washington, Counting spanning trees in the graphs of Kleitman and Golden and a generalization, Journal of the Franklin Institute 318 (1984) 349–355.
[13] X. Yong, T. Atajan, The numbers of spanning trees of the cubic cycle C_N^3 and the quadruple cycle C_N^4, Discrete Mathematics 169 (1-3) (1997) 293–298.
[14] R. Sedgewick, P. Flajolet, An Introduction to the Analysis of Algorithms, Addison-Wesley Publishing House, 1996 and Chebyshev polynomials, Graph and Combinatorics 2 (1986) 191–200.
[15] X. Yong, F.J. Zhang, An asymptotic behavior of the complexity of double fixed step loop networks, Applied Mathematics. A Journal of Chinese Universities. Ser. B 12 (1997) 233–236.
[16] F. Zhang, X. Yong, Asymptotic enumeration theorems for the numbers of spanning trees and Eulerian trails in circulant digraphs and graphs, Science in China, Ser. A 43 (1999) 264–271.