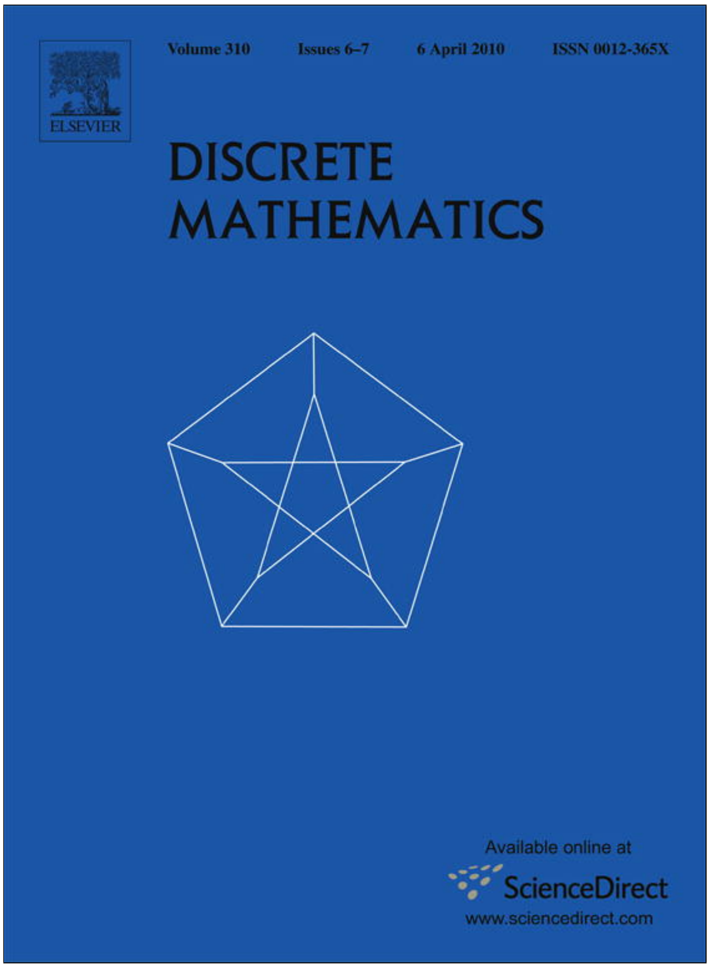
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An efficient approach for counting the number of spanning trees in circulant and related graphs

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ABSTRACT

The most recent general result for counting the exact number of spanning trees in a directed or an undirected circulant graph is that the numbers satisfy a recurrence relation of size 2^{s-1} where s is the largest jump [29]. A drawback here is that, when the jump s is large, it is difficult to apply the method to get the number of spanning trees because the degree of the recurrence relation grows exponentially and the coefficient matrix (it is an integral Toeplitz matrix of exponential size) of the linear system for establishing recurrence formula is not well conditioned in calculation.

In this paper, we focus our attention on this point and obtain an efficient approach (another kind of recursive formula) for counting the number of spanning trees in a directed or undirected circulant graph which has fixed or non-fixed jumps. The technique is also applied to the graphs $G = K_n \pm C$, where K_n is the complete graph on n vertices and C is a circulant graph. Compared with the previous approaches, our advantage is that, for any given jumps $s_1 < s_2 < \cdots < s_k$, the number of spanning trees can be calculated directly by a new kind of recursive formula, without establishing the recurrence relation of order 2^{s_k-1}. We describe our method by giving concrete examples of its use.

1. Introduction

Throughout this paper multiple graphs or digraphs are allowed to appear in our consideration. A spanning tree in an undirected graph G is a tree that has the same vertex set as G. An oriented spanning tree in a digraph D is a rooted tree with the same vertex set as D, that is, there is a node specified as the root and from it there is a path to any vertex of D. The study of the number of spanning trees in a graph has a long history and has been very active because, theoretically, counting the number is interesting, and the problem has different practical applications in different fields. For example, the number characterizes the reliability of a network and in physics, designing electrical circuits, analyzing energy of masers and investigating the possible particle transitions [8,10,14,15,21].

A well-known theoretical result on finding the number is the Matrix Tree Theorem [19] which expresses the number of spanning trees in terms of the determinant of a matrix obtained from the Laplace matrix of the graph. Unfortunately, counting the number by evaluating the determinant directly is hard for large graphs [1,4,32]. Due to this reason there have been developed techniques to get around the difficulties [5,11,12,25,29] and have paid much attention to deriving explicit and possibly simple formulas for certain special classes of graphs. For example, if G is the complete graph K_n, then Cayley's tree formula [18] states that $T(K_n) = n^{n-2}$. Some most recently derived results about the counting and maximizing the

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number of spanning trees can be found from [13,24,26,28,31]. Our interest here is to consider a class of special graphs, circulants.

Let $s_1, s_2, \ldots, s_k$ be given integers, called jumps, and $1 \le s_1 < s_2 < \cdots < s_k$. A circulant graph $C_n^{s_1,s_2,\ldots,s_k}$ has n vertices labeled $0, 1, 2, \ldots, n-1$, with each vertex i $(0 \le i \le n-1)$ adjacent to vertices $i \pm s_1, i \pm s_2, \ldots, i \pm s_k \pmod n$. A directed circulant graph, $\vec{C}_n^{s_1,s_2,\ldots,s_k}$, is a digraph on n vertices $0, 1, 2, \ldots, n-1$ and for each vertex i $(0 \le i \le n-1)$, there are k arcs from i to vertices $i + s_1, i + s_2, \ldots, i + s_k \pmod n$. Note that $C_n^{s_1,s_2,\ldots,s_k}$ is a $2k$-regular graph and that $\vec{C}_n^{s_1,s_2,\ldots,s_k}$ is a k-regular digraph. It is seen that the simplest undirected circulant graph is the n vertex cycle C_n^1 and that the simplest directed circulant graph is the n vertex directed cycle $\vec{C}_n^1$. As usual, we use $T(X)$ to stand for the number of spanning trees in a graph or digraph X. Starting from the different proofs [3,20,27] of the Conjecture $T(C_n^{1,2}) = nF_n^2$, where F_n the Fibonacci numbers, of Bedrosian [6] (which was also conjectured by Boesch and Wang in [9] without the knowledge of [20]), the recurrence formulas for $T(C_n^{1,3})$, $T(C_n^{1,4})$ were obtained in [27] and then more general results were recently obtained in [29,30]. The most general result states that $T(C_n^{s_1,s_2,\ldots,s_k}) = na_n^2$ where a_n satisfies a linear recurrence relation of order 2^{s_k-1}. For a directed circulant, the general formula is $T(\vec{C}_n^{s_1,s_2,\ldots,s_k}) = nb_n$, where b_n satisfies a linear recurrence relation of order 2^{s_k-1} [29]. There also have been combinatorial approaches on the number of spanning trees in odd-valiant circulants and some interesting combinatorial properties of a_n were derived in [2,11].

The formula for the number of spanning trees in the graph $K_n - S$ has been studied for different types of S. For instance, when S is a path or a cycle or a complete graph, similar results are established in [6]. A closed formula for $T(K_n - C_m^s)$ was obtained in [17] and the same formula was also reproved alternatively by introducing a different technique [30] and there is a generalization of their approach to getting a formulas for $T(K_n \pm C_m^{s_1,s_2,\ldots,s_k})$.

For a large s_k, it is hard to use the recurrence relation method to get the number of spanning trees in a circulant graph because the degree of the recurrence relation increases exponentially with s_k. In this paper, we focus our attention on this point and obtain another kind (recursive) formula for counting the number of spanning trees in a directed or undirected circulant graph with fixed or non-fixed jumps and formulas for $G = K_n \pm C$, where K_n is the complete graph on n vertices and C is a circulant graph. Our result states that, in finding the number of spanning trees, it is not necessary to solve a system of linear equations of exponential size (where the coefficient matrix of the system for establishing the recurrence formula) is a Toeplitz matrix which, in computation, is not well conditioned [16]. Our advantage is that, for any given jumps $s_1 < s_2 < \cdots < s_k$, the number of spanning trees can be calculated directly, without establishing the recurrence relation of order 2^{s_k-1}. To make the thing clearer, we describe our method by giving concrete examples of its use.

2. Basic lemmas

In this section, we will derive some fundamental results which are essential in finding the exact number of spanning trees. Our technique is by applying some basic facts from combinatorics and number theory.

Lemma 1 (*G. Pólya [23]*). *For any positive integers n and r, $r = 1, 2, \ldots, n-1$, let*

$$\mu(n) = \sum_{(r,n)=1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{n} r}. \quad \text{(Möbius function)} \tag{1}$$

Then

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ is divisible by a square (apart from 1)}, \\ (-1)^{\upsilon(n)} & \text{in all other cases}, \end{cases}$$

where $\upsilon(n)$ is the number of distinct prime factors of n and $\mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$ is the nth unit root.

For simplicity, throughout the paper, we say that a column vector C is the coefficient vector of a polynomial $f(x)$ if C is constructed from the coefficients of the polynomial $f(x)$ and its ith component is the coefficient of x^i (where the last component of C is the constant term of $f(x)$). X^t denotes the transpose of a matrix(or vector) X. We also use O_k to stand for the k dimensional zero column vector.

Lemma 2. *Let $f(x) = \alpha_p x^p + \alpha_{p-1}x^{p-1} + \cdots + \alpha_1 x + \alpha_0$, $\alpha_p = 1$, be a complex or real polynomial and C_{n-1} the coefficient vector of polynomial $\prod_{i=1}^{n-1} f(x^i)$. Then, for $k = 2, 3, \ldots, n-1$, C_k satisfies the following recurrence relation*

$$C_k = \left(\begin{bmatrix} C_{k-1} \\ O_{pk} \end{bmatrix} \begin{bmatrix} O_k \\ C_{k-1} \\ O_{(p-1)k} \end{bmatrix} \begin{bmatrix} O_{2k} \\ C_{k-1} \\ O_{(p-2)k} \end{bmatrix} \cdots \begin{bmatrix} O_{(p-1)k} \\ C_{k-1} \\ O_k \end{bmatrix} \begin{bmatrix} O_{pk} \\ C_{k-1} \end{bmatrix} \right) \begin{bmatrix} \alpha_p \\ \alpha_{p-1} \\ \vdots \\ \alpha_0 \end{bmatrix},$$

with the initial (column) vector $C_1 = (\alpha_p, \alpha_{p-1}, \ldots, \alpha_1, \alpha_0)^t$.

Proof. Without loss of generality, we may let

$$\prod_{i=1}^{k-1} f(x^i) = (x^{\frac{pk(k-1)}{2}}, x^{\frac{pk(k-1)}{2}-1}, \dots, x, 1)C_{k-1}. \tag{2}$$

Then we have

$$\begin{aligned}\prod_{i=1}^{k} f(x^i) &= f(x^k)\prod_{i=1}^{k-1} f(x^i)\\ &= \alpha_p x^{pk}(x^{\frac{pk(k-1)}{2}}, x^{\frac{pk(k-1)}{2}-1}, \dots, x, 1)C_{k-1} + \alpha_{p-1}x^{(p-1)k}(x^{\frac{pk(k-1)}{2}}, x^{\frac{pk(k-1)}{2}-1}, \dots, x, 1)C_{k-1}\\ &\quad \cdots + \alpha_0(x^{\frac{pk(k-1)}{2}}, x^{\frac{pk(k-1)}{2}-1}, \dots, x, 1)C_{k-1}\\ &= A\left(\begin{bmatrix} C_{k-1}\\ O_{pk}\end{bmatrix} \begin{bmatrix} O_k\\ C_{k-1}\\ O_{(p-1)k}\end{bmatrix} \begin{bmatrix} O_{2k}\\ C_{k-1}\\ O_{(p-2)k}\end{bmatrix} \cdots \begin{bmatrix} O_{(p-1)k}\\ C_{k-1}\\ O_k\end{bmatrix} \begin{bmatrix} O_{pk}\\ C_{k-1}\end{bmatrix}\right)\begin{bmatrix}\alpha_p\\ \alpha_{p-1}\\ \vdots\\ \alpha_0\end{bmatrix},\end{aligned}$$

where $A = (x^{\frac{pk(k+1)}{2}}, x^{\frac{pk(k+1)}{2}-1}, \dots, x, 1)$. This completes the proof of the lemma. □

To illustrate this we have:

Example 1. Let $f(x) = x^2 + 3x + 1$. We calculate C_2, the coefficients vector of the polynomial $\prod_{i=1}^{2} f(x^i)$. Since then $C_1 = (1, 3, 1)^t$, by Lemma 2 we have that

$$C_2 = \left(\begin{bmatrix} C_1\\ O_4\end{bmatrix} \begin{bmatrix} O_2\\ C_1\\ O_2\end{bmatrix} \begin{bmatrix} O_4\\ C_1\end{bmatrix}\right)\begin{bmatrix}1\\3\\1\end{bmatrix} = \left(\begin{bmatrix}1\\3\\1\\0\\0\\0\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\\3\\1\\0\\0\end{bmatrix} \begin{bmatrix}0\\0\\0\\0\\1\\3\\1\end{bmatrix}\right)\begin{bmatrix}1\\3\\1\end{bmatrix} = \begin{bmatrix}1\\3\\4\\9\\4\\3\\1\end{bmatrix}.$$

Note that in the above lemma if C_1 is symmetric (i.e., its ith component is equal to the $(p-i)$th component for all i) then all C_k are symmetric. The components of C_k also have the following properties.

Lemma 3. *Let $f(x) = \alpha_p x^p + \alpha_{p-1}x^{p-1} + \cdots + \alpha_1 x + \alpha_0$, $\alpha_p = 1$, be a complex or real polynomial, c_j, $j = pn(n-1)/2, \dots, 2, 1$ the components of the coefficient vector, C_{n-1}, of polynomial $\prod_{i=1}^{n-1} f(x^i)$ and*

$$w(k) = \sum_{j=k \pmod n} c_j. \tag{3}$$

Then for every $q_1, q_2 \in \{1, 2, 3, \dots, n-1\}$ we have that

$$w(q_1) = w(q_2) \quad \text{if } gcd(q_1, n) = gcd(q_2, n).$$

Proof. We assume that M is the companion matrix of $f(x)$, that is,

$$M = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -\alpha_0\\ 1 & 0 & 0 & \cdots & 0 & -\alpha_1\\ 0 & 1 & 0 & \cdots & 0 & -\alpha_2\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & -\alpha_{p-1}\end{pmatrix}.$$

Then the following two expressions are identical and we have their expansion (where I is the identity matrix)

$$\begin{aligned}\prod_{j=1}^{n-1} f(x^j) &= \prod_{j=1}^{n-1} |x^j I - M|\\ &= c_{\frac{pn(n-1)}{2}} x^{\frac{pn(n-1)}{2}} + c_{\frac{pn(n-1)}{2}-1} x^{\frac{pn(n-1)}{2}-1} + \cdots + c_1 x + c_0.\end{aligned}$$

Set $x = e^{\frac{2\pi i}{n}}$ and let $(1 =)n_0 < n_1 < \cdots < n_l(= n)$ be the all possible divisors of n. Then

$$\begin{aligned}\prod_{j=1}^{n-1} |e^{\frac{2\pi i}{n}j} I - M| &= c_{\frac{pn(n-1)}{2}} e^{\frac{2\pi i}{n}\frac{pn(n-1)}{2}} + c_{\frac{pn(n-1)}{2}-1} e^{\frac{2\pi i}{n}(\frac{pn(n-1)}{2}-1)} + \cdots + c_1 e^{\frac{2\pi i}{n}} + c_0 \\ &= w(0) + w(1)e^{\frac{2\pi i}{n}} + w(2)e^{\frac{2\pi i}{n}2} + \cdots + w(n-1)e^{\frac{2\pi i}{n}(n-1)} + c_0 \\ &= w(0) + \sum_{gcd(j,n)=n_0} w(j)e^{\frac{2\pi i}{n}j} + \cdots + \sum_{gcd(j,n)=n_{l-1}} w(j)e^{\frac{2\pi i}{n}j} + c_0.\end{aligned}$$

Note that for any pair of positive integers t' and t'', if $gcd(t', n) = gcd(t'', n) = 1$ then

$$\prod_{j=1}^{n-1} |e^{\frac{2\pi i}{n}t'j} I - M| = \prod_{j=1}^{n-1} |e^{\frac{2\pi i}{n}t''j} I - M|, \tag{4}$$

that is

$$\sum_{gcd(j,n)=n_0} w(j)e^{\frac{2\pi i}{n}t'j} + \cdots + \sum_{gcd(j,n)=n_{l-1}} w(j)e^{\frac{2\pi i}{n}t'j} = \sum_{gcd(j,n)=n_0} w(j)e^{\frac{2\pi i}{n}t''j} + \cdots + \sum_{gcd(j,n)=n_{l-1}} w(j)e^{\frac{2\pi i}{n}t''j}. \tag{5}$$

Now, for $m = 1, 2, \ldots, l-1$, writing each term of both sides in (5) as an inner product of a row vector W_m with a column vector $E_m(.)$, we have that

$$W_0E_0(t') + W_1E_1(t') + \cdots + W_{l-1}E_{l-1}(t') = W_0E_0(t'') + W_1E_1(t'') + \cdots + W_{l-1}E_{l-1}(t''), \tag{6}$$

where, for each m,

$$\begin{aligned}W_mE_m(t') &= \sum_{gcd(j,n)=n_m} w(j)e^{\frac{2\pi i}{n}t'j} \\ &= (w(j_1), w(j_2), \ldots, w(j_m))(e^{\frac{2\pi i}{n}t'j_1}, e^{\frac{2\pi i}{n}t'j_2}, \ldots, e^{\frac{2\pi i}{n}t'j_m})^t.\end{aligned}$$

Noting that $E_m(t')$ is a permutation of $E_m(t'')$, there exists a permutation matrix P_m such that $E_m(t'') = P_mE_m(t')$ and thus plugging them into (6) generates

$$W_0E_0(t') + \cdots + W_{l-1}E_{l-1}(t') = W_0P_0E_0(t') + \cdots + W_{l-1}P_{l-1}E_{l-1}(t'). \tag{7}$$

And this is true for any integer t' with $(t', n) = 1$. Therefore comparing the coefficient of $E_m(t')$ on both sides of (7), we have $W_m = W_mP_m$. This in turn implies that

$$w(j_1) = w(j_2) = \cdots = w(j_m), \tag{8}$$

completing the proof of the lemma. □

Lemma 4. *Let $f(x) = \alpha_p x^p + \alpha_{p-1}x^{p-1} + \cdots + \alpha_1 x + \alpha_0$, $\alpha_p = 1$, be a complex or real polynomial and $(1 =)n_0 < n_1 < \cdots < n_l(= n)$ be all possible divisors of n. Then we have*

$$\prod_{j=1}^{n-1} f(\varepsilon^j) = \left[\alpha_0^{n-1} + \sum_{j=0}^{l} \mu(n/n_j) \sum_{i=0}^{\lceil p(n-1)/2\rceil -1} c_{ni+n_j}\right], \tag{9}$$

where $\varepsilon = e^{\frac{2\pi i}{n}}$, c_j is the jth component of coefficient vector, C_{n-1}, of polynomial $\prod_{i=1}^{n-1} f(x^i)$ and for $k = 2, 3, \ldots, n-1$, C_k satisfies the following recurrence relation

$$C_k = \left(\begin{bmatrix} C_{k-1} \\ O_{pk} \end{bmatrix} \begin{bmatrix} O_k \\ C_{k-1} \\ O_{(p-1)k} \end{bmatrix} \begin{bmatrix} O_{2k} \\ C_{k-1} \\ O_{(p-2)k} \end{bmatrix} \cdots \begin{bmatrix} O_{(p-1)k} \\ C_{k-1} \\ O_k \end{bmatrix} \begin{bmatrix} O_{pk} \\ C_{k-1} \end{bmatrix} \right) \begin{bmatrix} \alpha_p \\ \alpha_{p-1} \\ \vdots \\ \alpha_0 \end{bmatrix},$$

with initial vector $C_1 = (\alpha_p, \alpha_{p-1}, \alpha_{p-2}, \ldots, \alpha_1, \alpha_0)^t$.

Proof. From Lemmas 2 and 3 and their proofs, it is easily obtained that $c_0 = \alpha_0^{n-1}$, $w(0) = w(n)$ and that

$$\prod_{j=1}^{n-1} f(\varepsilon^j) = \alpha_0^{n-1} + \sum_{gcd(j,n)=n_0} w(j)e^{\frac{2\pi i}{n}j} + \sum_{gcd(j,n)=n_1} w(j)e^{\frac{2\pi i}{n}j} + \cdots + \sum_{gcd(j,n)=n_l} w(j)e^{\frac{2\pi i}{n}j}.$$

Noticing that, if n_d is a divisor of n then $n_d \in \{j \mid gcd(j, n) = n_d,\ j = 1, 2, \dots, n\}$, we have

$$\prod_{j=1}^{n-1} f(\varepsilon^j) = \alpha_0^{n-1} + w(n_0) \sum_{gcd(j,n)=n_0} \mathrm{e}^{\frac{2\pi \mathrm{i}}{n} j} + w(n_1) \sum_{gcd(j,n)=n_1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{n} j} + \cdots + w(n_l) \sum_{gcd(j,n)=n_l} \mathrm{e}^{\frac{2\pi \mathrm{i}}{n} j}.$$

Further let $n/n_d = p_d, j/n_d = r_d$. Then the above formula is reduced to

$$\prod_{j=1}^{n-1} f(\varepsilon^j) = \alpha_0^{n-1} + w(n_0) \sum_{gcd(r_0,p_0)=1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{p_0} r_0} + w(n_1) \sum_{gcd(r_1,p_1)=1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{p_1} r_1} + \cdots + w(n_l) \sum_{gcd(r_l,p_l)=1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{p_l} r_l}.$$

Thus, for $d = 0, 1, 2, \dots, l$, applying Lemma 1 yields

$$\begin{aligned}\sum_{gcd(r_d,p_d)=1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{p_d} r_d} &= \mu(p_d)\\ &= \mu(n/n_d),\end{aligned}$$

and from Lemma 3, the components of $C_{n-1}, c_j, j = pn(n-1)/2, \dots, 2, 1$ satisfy

$$\begin{aligned}w(n_d) &= \sum_{j=n_d (mod\ n)} c_j\\ &= \sum_{j=0}^{\lceil p(n-1)/2 \rceil - 1} c_{nj+n_d}.\end{aligned}$$

The proof of the lemma is thus completed. □

Lemma 5. *If $f(x)$ is a reciprocal polynomial of degree p then the product $\prod_{i=1}^{n-1} f(x^i)$ is a also reciprocal polynomial.*

Proof. If $f(x)$ is a reciprocal polynomial of degree p [22], that is $f(x) = x^p f(x^{-1})$. Thus,

$$\begin{aligned}\prod_{i=1}^{n-1} f(x^i) &= \prod_{i=1}^{n-1} f(x^i)\\ &= \prod_{i=1}^{n-1} x^{pi} f(x^{-i})\\ &= x^{pn(n-1)/2} \prod_{i=1}^{n-1} f(x^{-i}),\end{aligned}$$

and this proves the lemma. □

3. Spanning trees in circulant graphs

In this section we derive the formulas for finding the exact numbers of spanning trees in circulants with fixed and non-fixed jumps. The idea is by making use of the results obtained in the previous section.

Lemma 6 (*Biggs [7], and Zhang and Yong [32]*). *For any integer $1 \le s_1 < s_2 < \cdots < s_k \le \lfloor \frac{n}{2} \rfloor$*

$$T(C_n^{s_1,s_2,\dots,s_k}) = \frac{1}{n} \prod_{i=1}^{n-1} (2k - \varepsilon^{-s_1 j} - \varepsilon^{s_1 j} - \varepsilon^{-s_2 j} - \varepsilon^{s_2 j} - \cdots - \varepsilon^{-s_k j} - \varepsilon^{s_k j}), \tag{10}$$

$$T(\vec{C}_n^{s_1,s_2,\dots,s_k}) = \prod_{i=1}^{n-1} (k - \varepsilon^{s_1 j} - \varepsilon^{s_2 j} - \cdots - \varepsilon^{s_k j}), \tag{11}$$

where ε^{-j} is the conjugate of ε^j, $\varepsilon = \mathrm{e}^{2\pi \mathrm{i}/n}$.

For convenience, let

$$f_{s_1,s_2,\dots,s_k}(x) = x^{2s_k} + x^{s_k+s_{k-1}} \cdots + x^{s_k+s_1} - 2kx^{s_k} + x^{s_k-s_1} + \cdots + x^{s_k-s_{k-1}} + 1, \tag{12}$$

$$\vec{f}_{s_1,s_2,\dots,s_k}(x) = x^{s_k} + x^{s_{k-1}} + \cdots + x^{s_1} - k. \tag{13}$$

Then, counting the number of spanning trees of an undirected circulant graph can be formulated as

Theorem 7. *For $1 \leq s_1 < s_2 < \cdots < s_k \leq \lfloor n/2 \rfloor$, let $(1 =)n_0 < n_1 < \cdots < n_l(= n)$ be the all possible divisors of n. Then we have*

$$T(C_n^{s_1,s_2,\ldots,s_k}) = \frac{(-1)^{(s_k+1)(n-1)}}{n}\left[1 + \sum_{j=0}^{l} \mu(n/n_j) \sum_{i=0}^{s_k(n-1)-1} c_{in+n_j}\right], \tag{14}$$

where c_j is the jth component of coefficient vector, C_{n-1}, of polynomial $\prod_{i=1}^{n-1} f_{s_1,s_2,\ldots,s_k}(x^i)$ and for $m = 2, 3, \ldots, n-1$, C_m satisfies the following recurrence relation

$$C_m = \left(\begin{bmatrix} C_{m-1} \\ O_{2s_km} \end{bmatrix} \quad \begin{bmatrix} O_{(s_k-s_{k-1})m} \\ C_{m-1} \\ O_{(s_k+s_{k-1})m} \end{bmatrix} \quad \begin{bmatrix} O_{(s_k-s_{k-2})m} \\ C_{m-1} \\ O_{(s_k+s_{k-2})m} \end{bmatrix} \quad \cdots \quad \begin{bmatrix} O_{2s_km} \\ C_{m-1} \end{bmatrix}\right) \begin{bmatrix} 1 \\ \vdots \\ 1 \\ -2k \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

with the initial vector C_1 (the coefficient vector of the polynomial $f_{s_1,s_2,\ldots,s_k}(x)$).

Proof. From (10) and (12) we have that

$$\begin{aligned} T(C_n^{s_1,s_2,\ldots,s_k}) &= \frac{1}{n}\prod_{j=1}^{n-1}(2k - \varepsilon^{-s_1j} - \varepsilon^{s_1j} - \varepsilon^{-s_2j} - \varepsilon^{s_2j} - \cdots - \varepsilon^{-s_kj} - \varepsilon^{s_kj}) \\ &= \frac{1}{n}\prod_{j=1}^{n-1} -\varepsilon^{-s_kj}\prod_{j=1}^{n-1}(\varepsilon^{2s_kj} + \cdots + \varepsilon^{(s_k+s_1)j} - 2k\varepsilon^{s_kj} + \varepsilon^{(s_k-s_1)j} + \cdots + 1) \\ &= \frac{(-1)^{(s_k+1)(n-1)}}{n}\prod_{j=1}^{n-1}(\varepsilon^{2s_kj} + \cdots + \varepsilon^{(s_k+s_1)j} - 2k\varepsilon^{s_kj} + \varepsilon^{(s_k-s_1)j} + \cdots + 1) \\ &= \frac{(-1)^{(s_k+1)(n-1)}}{n}\prod_{j=1}^{n-1} f_{s_1,\ldots,s_k}(\varepsilon^j). \end{aligned}$$

The remaining arguments are analogous to those in Lemma 4. We therefore omit the details here. □

We now illustrate our technique by finding the number of spanning trees in a simple graph.

Example 2. $T(C_4^{1,2})$.

In this case $s_1 = 1, s_2 = 2, f(x) = x^4 + x^3 - 4x^2 + x + 1, C_1 = (1,\ 1,\ -4,\ 1,\ 1)^t$, $n = 4$ and the all possible divisors of n are 1, 2 and 4. So, from (14) we have that

$$T(C_4^{1,2}) = \frac{-1}{4}\left[1 + \mu(1)\sum_{i=0}^{5} c_{4i+4} + \mu(2)\sum_{i=0}^{5} c_{4i+2} + \mu(4)\sum_{i=0}^{5} c_{4i+1}\right], \tag{15}$$

where c_j is the jth component of the coefficient vector C_3, it can be easily obtained by the following recurrence relation

$$C_m = \left(\begin{bmatrix} C_{m-1} \\ O_{4m} \end{bmatrix} \quad \begin{bmatrix} O_m \\ C_{m-1} \\ O_{3m} \end{bmatrix} \quad \begin{bmatrix} O_{2m} \\ C_{m-1} \\ O_{2m} \end{bmatrix} \quad \begin{bmatrix} O_{3m} \\ C_{m-1} \\ O_m \end{bmatrix} \quad \begin{bmatrix} O_{4m} \\ C_{m-1} \end{bmatrix}\right) \begin{bmatrix} 1 \\ 1 \\ -4 \\ 1 \\ 1 \end{bmatrix}, \tag{16}$$

recursive calculation of C_m for $m = 2, 3$ yields (note that C_k are symmetric)

$$C_3 = (1, 1, -3, 3, -6, -6, 16, -14, 2, 13, 23, 3, -66, 3, 23, 13, 2, -14, 16, -6, -6, 3, -3, 1, 1)^t.$$

On the other hand, from Lemma 1 $\mu(4) = 0$, $\mu(2) = -1$, $\mu(1) = 1$. So plunging them into the formula gives

$$\begin{aligned} T(C_4^{1,2}) &= -\frac{1}{4}\left[1 + \sum_{i=0}^{5} c_{4i+4} - \sum_{i=0}^{5} c_{4i+2}\right] \\ &= -\frac{1}{4}[1 - 73 - 72] \\ &= 36 \\ &= 4 \times 3^2. \end{aligned}$$

In the Introduction we noted that $T(C_n^{1,2}) = nF_n^2$, where F_n is the well-known Fibonacci numbers, this gives $T(C_4^{1,2}) = 4F_4^2 = 4 \times 3^2$. The result is the same as the one we obtained in Example 2.

A similar derivation yields the following theorem about counting the number of spanning trees in a directed circulant graph.

Theorem 8. *For $1 \le s_1 < s_2 < \cdots < s_k \le n-1$, let $(1 =)n_0 < n_1 < \cdots < n_l (= n)$ be the all possible divisors of n. Then we have*

$$T(\vec{C}_n^{s_1,s_2,\ldots,s_k}) = k^{n-1} + (-1)^{n-1} \sum_{j=0}^{l} \mu(n/n_j) \sum_{i=0}^{\lceil s_k(n-1)/2\rceil - 1} c_{in+n_j}, \tag{17}$$

where c_j is the jth component of the coefficient vector, C_{n-1}, of polynomial $\prod_{i=1}^{n-1} \vec{f}_{s_1,s_2,\ldots,s_k}(x^i)$ and for $m = 2, 3, \ldots, n-1$, C_m satisfies the following recurrence relation

$$C_m = \left(\begin{bmatrix} C_{m-1} \\ O_{s_k m}\end{bmatrix} \quad \begin{bmatrix} O_{(s_k - s_{k-1})m} \\ C_{m-1} \\ O_{s_{k-1}m}\end{bmatrix} \quad \begin{bmatrix} O_{(s_k - s_{k-2})m} \\ C_{m-1} \\ O_{s_{k-2}m}\end{bmatrix} \quad \cdots \quad \begin{bmatrix} O_{s_k m} \\ C_{m-1}\end{bmatrix}\right) \begin{bmatrix} 1 \\ \vdots \\ 1 \\ -k \end{bmatrix},$$

with the initial vector C_1 which is the coefficient vector of polynomial $\vec{f}_{s_1,s_2,\ldots,s_k}(x)$.

Proof. The proof is similar to that of Theorem 7 and we therefore omit the details. □

Below is an example for a directed graph.

Example 3. $T(\vec{C}_4^{1,2})$.

In this case we have $s_1 = 1, s_2 = 2, f(x) = x^2 + x - 2, C_1 = (1,\ 1,\ -2)^t$, $n = 4$ and all possible divisors of n are 1, 2 and 4. So, from (17) we have

$$T(\vec{C}_4^{1,2}) = 2^3 - \left[\mu(1)\sum_{i=0}^{2} c_{4i+4} + \mu(2)\sum_{i=0}^{2} c_{4i+2} + \mu(4)\sum_{i=0}^{2} c_{4i+1}\right], \tag{18}$$

where c_j is the jth component of the coefficient vector C_3, it can be easily obtained by the following recurrence relation:

$$C_m = \left(\begin{bmatrix} C_{m-1} \\ O_{2m}\end{bmatrix} \quad \begin{bmatrix} O_m \\ C_{m-1} \\ O_m\end{bmatrix} \quad \begin{bmatrix} O_{2m} \\ C_{m-1}\end{bmatrix}\right) \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}. \tag{19}$$

Recursive calculation gives $C_3 = (1, 1, -1, 2, -3, -3, 3, -6, 0, 2, 8, 4, -8)^t$. From Lemma 1, $\mu(4) = 0$, $\mu(2) = -1$, $\mu(1) = 1$, plugging them into the formula we have

$$\begin{aligned} T(\vec{C}_4^{1,2}) &= 2^3 - \left[\sum_{i=0}^{2} c_{4i+4} - \sum_{i=0}^{2} c_{4i+2}\right] \\ &= 8 - [-2 - 10] \\ &= 20. \end{aligned}$$

Note that the result is the same as the one in Example 3 of [27] (where it is given as $8 + 4 + 8 = 20$).

Noticing further from Theorems 7 and 8 we know that if the number of divisors of n is small then the formulas (14) and (17) will become simpler. When n is a prime number, we have the following two corollaries.

Corollary 1. *Let $1 \le s_1 < s_2 < \cdots < s_k \le \lfloor n/2 \rfloor$, and n is a prime number. Then we have*

$$T(C_n^{s_1,s_2,\ldots,s_k}) = \frac{(-1)^{(s_k+1)(n-1)}}{n}\left[1 + \sum_{i=0}^{s_k(n-1)-1}(c_{in+n} - c_{in+1})\right], \tag{20}$$

where c_j is the jth component of the coefficient vector, C_{n-1}, of polynomial $\prod_{i=1}^{n-1} f_{s_1,s_2,\ldots,s_k}(x^i)$ and for $m = 2, 3, \ldots, n-1$, C_m satisfies the following recurrence relation

$$C_m = \left(\begin{bmatrix} C_{m-1} \\ O_{2s_km} \end{bmatrix} \quad \begin{bmatrix} O_{(s_k-s_{k-1})m} \\ C_{m-1} \\ O_{(s_k+s_{k-1})m} \end{bmatrix} \quad \begin{bmatrix} O_{(s_k-s_{k-2})m} \\ C_{m-1} \\ O_{(s_k+s_{k-2})m} \end{bmatrix} \quad \cdots \quad \begin{bmatrix} O_{2s_km} \\ C_{m-1} \end{bmatrix}\right) \begin{bmatrix} 1 \\ \vdots \\ 1 \\ -2k \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

with the initial vector C_1 which is the coefficient vector of the polynomial $f_{s_1,s_2,\ldots,s_k}(x)$.

Proof. The divisors of n are 1 and n itself because n is a prime number. The rest of the proof follows that as in Theorem 7. □

Example 4. $T(C_{47}^{5,16})$.

In this case $s_1 = 5, s_2 = 16, f(x) = x^{32} + x^{21} - 4x^{16} + x^{11} + 1, n = 47$ and all possible divisors of n are 1 and 47. So from Corollary 1 we have

$$T(C_{47}^{5,16}) = \frac{1}{47}\left[1 + \sum_{i=0}^{735}(c_{47i+47} - c_{47i+1})\right], \tag{21}$$

where c_j is the jth component of the coefficient vector C_{46}, it can be easily obtained by the following recurrence relation

$$C_m = \left(\begin{bmatrix} C_{m-1} \\ O_{32m} \end{bmatrix} \quad \begin{bmatrix} O_{11m} \\ C_{m-1} \\ O_{21m} \end{bmatrix} \quad \begin{bmatrix} O_{16m} \\ C_{m-1} \\ O_{16m} \end{bmatrix} \quad \begin{bmatrix} O_{21m} \\ C_{m-1} \\ O_{11m} \end{bmatrix} \quad \begin{bmatrix} O_{32m} \\ C_{m-1} \end{bmatrix}\right) \begin{bmatrix} 1 \\ 1 \\ -4 \\ 1 \\ 1 \end{bmatrix}, \tag{22}$$

with the initial vector

$$C_1 = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -4, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)^t.$$

Then recursive calculation for $m = 2, 3, \ldots, 46$ yields the following result

$$\begin{aligned} T(C_{47}^{5,16}) &= \frac{1}{47}262636412799607294967 5369 \\ &= 5588008782970367978 0327 \\ &= 47 \times 118893803892986552 7241 \\ &= 47 \times 34480980829^2. \end{aligned}$$

Remark 1. We would like to point out that this method is much more efficient than the one introduced in [29] because, in that method, we need to establish the recurrence formula of order $2^{16-1} = 32768$ and, to this end, we attempt to solve a linear system of size 32768×32768 apart from finding the initial values. Exact calculation will therefore become hard!

Corollary 2. *Let $1 \le s_1 < s_2 < \cdots < s_k \le n-1$, and n is a prime number. Then we have*

$$T(\vec{C}_n^{s_1,s_2,\ldots,s_k}) = k^{n-1} + (-1)^{n-1}\sum_{i=0}^{\lceil s_k(n-1)/2\rceil - 1}(c_{in+n} - c_{in+1}), \tag{23}$$

where c_j is the jth component of the coefficient vector, C_{n-1}, of polynomial $\prod_{i=1}^{n-1}\vec{f}_{s_1,s_2,\ldots,s_k}(x^i)$ and for $m = 2, 3, \ldots, n-1$, C_m satisfies the following recursive formula

$$C_m = \left(\begin{bmatrix} C_{m-1} \\ O_{s_k m} \end{bmatrix} \quad \begin{bmatrix} O_{(s_k-s_{k-1})m} \\ C_{m-1} \\ O_{s_{k-1}m} \end{bmatrix} \quad \begin{bmatrix} O_{(s_k-s_{k-2})m} \\ C_{m-1} \\ O_{s_{k-2}m} \end{bmatrix} \quad \cdots \quad \begin{bmatrix} O_{s_k m} \\ C_{m-1} \end{bmatrix}\right) \begin{bmatrix} 1 \\ \vdots \\ 1 \\ -k \end{bmatrix},$$

with the initial vector C_1, the coefficient vector of the polynomial $\vec{f}_{s_1,s_2,\ldots,s_k}(x)$.

Proof. The proof of this corollary is similar to that of Theorem 8. □

4. The number of spanning trees of circulant related graphs

In this section, we consider the formulas for the number of spanning trees of graphs $K_n \pm C$ where C are circulants. To this end we need the following known lemmas.

Lemma 9 (*Y.P. Zhang, X. Yong, M.J. Golin [30]*). *For $n \geq \sum_{u=1}^{l} m_u$ and for each u, $1 \leq u \leq l$, $m_u > 2s_{k_u}$,*

$$T\left(K_n - \bigcup_{u=1}^{l} C_{m_u}^{s_{1u},s_{2u},\ldots,s_{ku}}\right) = n^{n-\sum\limits_{u=1}^{l} m_u + l - 2} \prod_{u=1}^{l}\prod_{j=1}^{m_u-1}(n - 2k_u + \varepsilon^{-s_{1u}j} + \varepsilon^{s_{1u}j} + \cdots + \varepsilon^{-s_{ku}j} + \varepsilon^{s_{ku}j}),$$

where $\varepsilon_u = \mathrm{e}^{\frac{2\pi\mathrm{i}}{m_u}}$.

Lemma 10 (*Y.P. Zhang, X. Yong, M.J. Golin [30]*). *For $n \geq \sum_{u=1}^{l} m_u$ and for each u, $1 \leq u \leq l$, $m_u > 2s_{k_u}$,*

$$T\left(K_n + \bigcup_{u=1}^{l} C_{m_u}^{s_{1u},s_{2u},\ldots,s_{ku}}\right) = n^{n-\sum\limits_{u=1}^{l} m_u + l - 2} \prod_{u=1}^{l}\prod_{j=1}^{m_u-1}(n + 2k_u - \varepsilon^{-s_{1u}j} - \varepsilon^{s_{1u}j} - \cdots - \varepsilon^{-s_{ku}j} - \varepsilon^{s_{ku}j}),$$

where $\varepsilon_u = \mathrm{e}^{\frac{2\pi\mathrm{i}}{m_u}}$, and for each u, $1 \leq u \leq l$.

For simplicity, we write the above two formulas, $T(K_n - \bigcup_{u=1}^{l} C_{m_u}^{s_{1u},s_{2u},\ldots,s_{ku}})$ and $T(K_n + \bigcup_{u=1}^{l} C_{m_u}^{s_{1u},s_{2u},\ldots,s_{ku}})$, into one expression $T(K_n - \delta\bigcup_{u=1}^{l} C_{m_u}^{s_{1u},s_{2u},\ldots,s_{ku}})$. Then when $\delta = 1$ it gives $T(K_n - \bigcup_{u=1}^{l} C_{m_u}^{s_{1u},s_{2u},\ldots,s_{ku}})$ and when $\delta = -1$ we have $T(K_n + \bigcup_{u=1}^{l} C_{m_u}^{s_{1u},s_{2u},\ldots,s_{ku}})$. We let

$$f_u(x) = x^{2s_{ku}} + x^{s_{ku}+s_{(k-1)u}} + \cdots + (\delta n - 2k_u)x^{s_{ku}} + x^{s_{ku}-s_{1u}} + \cdots + 1,$$

$$\Phi = \frac{n^{n-\sum\limits_{u=1}^{l} m_u + l - 2}\,\delta^{l(m_u-1)}}{\prod\limits_{u=1}^{l}(-1)^{s_{ku}(m_u-1)}}.$$

Then we have the following theorem.

Theorem 11. *For $1 \leq s_{1u} < s_{2u} < \cdots < s_{ku} \leq \lfloor m_u/2 \rfloor$, $n \geq \sum_{u=1}^{l} m_u$ and for each u, $1 \leq u \leq l$, let $(1 =)m_{0u}$, $m_{1u}, \ldots, m_{tu}(= m_u)$ be all possible divisors of m_u. Then we have*

$$T\left(K_n - \delta\bigcup_{u=1}^{l} C_{m_u}^{s_{1u},s_{2u},\ldots,s_{ku}}\right) = \Phi\prod_{u=1}^{l}\left[1 + \sum_{j=0}^{t}\mu(m_u/m_{ju})\sum_{i=0}^{s_{ku}(m_u-1)-1} c^{(u)}_{im_u+m_{ju}}\right], \tag{24}$$

where $c_j^{(u)}$ is the jth component of the coefficient vector, $C^{(u)}_{m_u-1}$, of polynomial $\prod_{i=1}^{m_u-1} f_u(x^i)$ and for $k = 2, 3, \ldots, m_u - 1$, $C_k^{(u)}$ satisfy the following recurrence relation

$$C_k^{(u)} = \left(\begin{bmatrix} C_{k-1}^{(u)} \\ O_{2s_{ku}k} \end{bmatrix} \quad \begin{bmatrix} O_{(s_{ku}-s_{(k-1)u})k} \\ C_{k-1}^{(u)} \\ O_{(s_{ku}+s_{(k-1)u})k} \end{bmatrix} \quad \begin{bmatrix} O_{(s_{ku}-s_{(k-2)u})k} \\ C_{k-1}^{(u)} \\ O_{(s_{ku}+s_{(k-2)u})k} \end{bmatrix} \quad \cdots \quad \begin{bmatrix} O_{2s_{ku}k} \\ C_{k-1}^{(u)} \end{bmatrix}\right) \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \delta n - 2k_u \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

with the initial vector $C_1^{(u)}$ which is the coefficient vector of the polynomial $f_u(x)$.

Proof.

$$T\left(K_n - \delta\bigcup_{u=1}^{l} C_{m_u}^{s_{1u},\ldots,s_{ku}}\right) = \frac{n^{n-\sum_{u=1}^{l} m_u+l-2}}{\delta^{l(m_u-1)}} \prod_{u=1}^{l}\prod_{j=1}^{m_u-1}(\delta n - 2k_u + \varepsilon^{-s_{1u}j} + \varepsilon^{s_{1u}j} + \cdots + \varepsilon^{-s_{ku}j} + \varepsilon^{s_{ku}j})$$

$$= \frac{n^{n-\sum_{u=1}^{l} m_u+l-2}\delta^{l(m_u-1)}}{\prod_{u=1}^{l}(-1)^{s_{ku}(m_u-1)}} \prod_{u=1}^{l}\prod_{j=1}^{m_u-1}(\varepsilon^{2s_{ku}j} + \varepsilon^{(s_{ku}+s_{(k-1)u})j} + \cdots + (\delta n - 2k_u)\varepsilon^{s_{ku}j} + \varepsilon^{(s_{ku}-s_{1u})j} + \cdots + 1)$$

$$= \Phi \prod_{u=1}^{l}\prod_{j=1}^{m_u-1} f_u(\varepsilon^j).$$

The remaining part of the proof is similar to that of Theorem 7. □

As a simple application of Theorem 11, we consider the following Example 5 (This is the combination of Corollaries 4, 5, 7, 8 of [30]).

Example 5.

(a) $T(K_n - \delta C_3^{1,2}) = n^{n-4}(v-2)^2$,

(b) $T(K_n - \delta C_4^{1,2}) = \delta n^{n-5} v(v-2)^2$,

(c) $T(K_n - \delta C_5^{1,2}) = n^{n-6}(v-1)^4$,

(d) $T(K_n - \delta C_6^{1,2}) = \delta n^{n-7} v^3 (v-2)^2$,

(e) $T(K_n - \delta C_7^{1,2}) = n^{n-8}(v^6 - 4v^5 + 2v^4 + 6v^3 - 3v^2 - v + 1)$,

where $v = \delta n - 4$, and $\delta \in \{-1, 1\}$.

Proof. Case (a): In this case we have that $l = 1, s_{11} = 1, s_{21} = 2, f_1(x) = x^4 + x^3 + vx^2 + x + 1, v = (\delta n - 4)$, $C_1^{(1)} = (1, 1, v, 1, 1,)^t$, $m_1 = 3$ and all possible divisors of m_1 are 1 and 3. So, from (24) we have

$$\begin{aligned} T(K_n - \delta C_3^{1,2}) &= \frac{n^{n-3+1-2}\delta^{(3-1)}}{(-1)^{2(3-1)}}\left[1 + \mu(3/3)\sum_{i=0}^{3}(c_{3i+3}^{(1)}) + \mu(3/1)\sum_{i=0}^{3}(c_{3i+1}^{(1)})\right] \\ &= n^{n-4}\left[1 + \sum_{i=0}^{3}(c_{3i+3}^{(1)}) - \sum_{i=0}^{3}(c_{3i+1}^{(1)})\right] \\ &= n^{n-4}(1 + c_3^{(1)} + c_6^{(1)} + c_9^{(1)} + c_{12}^{(1)} - c_1^{(1)} - c_4^{(1)} - c_7^{(1)} - c_{10}^{(1)}), \end{aligned}$$

where $c_j^{(1)}$ is the jth component of the coefficient vector, $C_2^{(1)}$, of polynomial $f_1(x)f_1(x^2)$ and

$$C_2^{(1)} = \left(\begin{bmatrix} C_1^{(1)} \\ O_8 \end{bmatrix} \begin{bmatrix} O_2 \\ C_1^{(1)} \\ O_6 \end{bmatrix} \begin{bmatrix} O_4 \\ C_1^{(1)} \\ O_4 \end{bmatrix} \begin{bmatrix} O_6 \\ C_1^{(1)} \\ O_2 \end{bmatrix} \begin{bmatrix} O_8 \\ C_1^{(1)} \end{bmatrix}\right) \begin{bmatrix} 1 \\ 1 \\ v \\ 1 \\ 1 \end{bmatrix}. \tag{25}$$

Simple calculation yields $c_3^{(1)} = 2, c_6^{(1)} = 2 + v, c_9^{(1)} = 2, c_{12}^{(1)} = 1, c_1^{(1)} = 1, c_4^{(1)} = 1 + 2v, c_7^{(1)} = 1 + v, c_{10}^{(1)} = 1 + v$. Plugging them into the above formula of $T(K_n - \delta C_3^{1,2})$ gives rise to

$$\begin{aligned} T(K_n - \delta C_3^{1,2}) &= n^{n-4}(v^2 - 4v + 4) \\ &= n^{n-4}(v-2)^2. \end{aligned}$$

This proves (a). The proofs of (b), (c), (d) and (e) are similar. So we omit them here. □

From Example 5 we know that $v = \delta n - 4$ and $\delta \in \{1, -1\}$, thus

$$T(K_n - \delta C_3^{1,2}) = n^{n-4}(\delta n - 6)^2, \tag{26}$$

and

$$T(K_n - C_3^{1,2}) = n^{n-4}(n-6)^2, \quad (\delta = 1), \tag{27}$$

$$T(K_n + C_3^{1,2}) = n^{n-4}(n+6)^2, \quad (\delta = -1). \tag{28}$$

Formula (28) is the same as the result of the first term of Corollary 8 in [30]. Here we obtain it in an alternative way.

Theorem 12. *For $1 \le s_1 < s_k < \cdots < s_k$, $\delta \in \{1, -1\}$ and $n \ge m$, let $v = \delta n - 2k$ and*

$$g(x) = \prod_{j=1}^{m-1}(x + \varepsilon^{-s_1 j} + \varepsilon^{s_1 j} + \varepsilon^{-s_2 j} + \varepsilon^{s_2 j} + \cdots + \varepsilon^{-s_k j} + \varepsilon^{s_k j}). \tag{29}$$

Then we have

(a) *$g(x)$ is an integer coefficient polynomial of degree $m-1$,*
(b) $(K_n - \delta C_m^{s_1,s_2,\ldots,s_k}) = \delta^{m-1} n^{n-m-1} g(v)$.

Proof. From the proof of Theorem 11, we know that each $c_i^{(u)}$ in (24) is an integer coefficient polynomial of $\delta n - 2k$. □

Remark 2. The exact calculation of the number of spanning trees, $T(C_n^{s_1,s_2,\ldots,s_k}) = na_n^2$, is the essential in network reliability analysis [9,21]. For any integer $s_1 < s_2 < \cdots < s_k$, to find the recurrence relations of a_n, both Theorem 8 of [29] and Theorem 3.1 of [12] need to calculate $2^{sk}(2^{sk-1})$ values of a_n and then solve a system of $2^{sk-1}(2^{sk-2})$ linear equations with unsymmetric Toeplitz matrix. Because of the exponential size, it is hard to solve such an unsymmetric Toeplitz system for a large s_k and the stability of the process cannot be assured unless its leading principle submatrices are sufficiently well conditioned [16]. Theorem 6 of [2] claims that it is not necessary to solve such a system of linear equations, but one still has to calculate $2^{sk}(2^{sk-1})$ values of a_n. Lemma 6 gives very good expressions of the number of spanning trees, but, unfortunately, we cannot use it in practice, since it cannot give the exact number even for small s_k and n. Theorem 7 gives the exact calculation formula that it is not necessary to calculate $2^{sk}(2^{sk-1})$ values of a_n or to solve a system of $2^{sk-1}(2^{sk-2})$ linear equations.

5. Concluding remarks

For a large s_k, it is not easy to apply the recurrence relation technique derived in [29] for finding the number of spanning trees in circulant graphs because the degree of the recurrence relation increases exponentially with s_k. For instance, to evaluate $T(C_n^{1,20})$ we have to establish a recurrence relation of order $2^{19}(= 524288)$ to get the number of spanning trees. This is hard!

Focusing our attention on this point, in this paper we obtained direct (recursive) formulas for the numbers of spanning trees in G where G is either a directed or an undirected circulant graph with fixed or non-fixed jumps. And for $G = K_n \pm C$, where K_n is the complete graph on n vertices and C is a circulant graph with jumps $s_1 < s_2 < \cdots < s_k$, the numbers can also be calculated efficiently. Our method introduced here does not need to establish the recurrence relation of order 2^{s_k-1}. An interesting problem would be to consider the *max* and the *min* value of the average growth rate of the number of spanning trees by (possibly) making use of the results obtained in this paper.

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