On the distribution of eigenvalues of graphs

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Abstract
Let G be a simple graph with n(≥ 2) vertices, and $\lambda_i(G)$ be the ith largest eigenvalue of G. In this paper we obtain the following:
If $\lambda_3(G) < 0$, and there exists some index k, $2 \leq k \leq [\frac{n}{2}]$,such that $\lambda_k(G) =$ -1, then

$$\lambda_j(G) = -1, \quad j = k, k+1, \cdots, n-k+1.$$

In particular, we obtain that (1) $\lambda_2(G) = -1$ implies

$$\lambda_1(G) = n-1, \quad \lambda_j(G) = -1, \quad j = 2, 3, \cdots, n.$$

and therefore G is complete. This is a result presented in [6]; (2) $\lambda_3(G) = -1$ implies that $\lambda_j(G) = -1, \quad j = 3, 4, \cdots, n-2$.

1.Introduction.

All graphs considered here are undirected and simple.

Let G denote a graph with vertex set $\{v_1, v_2, \cdots, v_n\}$. Its adjacency matrix $A(G)$ is the $n \times n$ one-zero matrix (a_{ij}), where a_{ij}=1 iff v_i is adjacent to v_j, and a_{ij}=0 otherwise. It is seen that $A(G)$ is a symmetric (0,1) matrix with every diagonal entry equal to zero. We shall denote the characteristic polynomial of G by

$$P(x, G) = det(xI - A(G)) = \sum_{i=0}^{n} a_i x^{n-i}.$$

Since $A(G)$ is a real symmetric matrix, its eigenvalues, say $\lambda_i(A(G))(i = 1, 2, \cdots, n)$, are real numbers, and may be ordered as $\lambda_1(A(G)) \geq \lambda_2(A(G)) \geq \cdots \geq \lambda_n(A(G))$. Denote $\lambda_i(A(G))$ simply by $\lambda_i(G)$. The sequence of n eigenvalues of G is known as the spectrum of G. Spectra of graphs appear frequently in the mathematical sciences. A good survey on this field can be found in [1].

The problem how to characterize a graph by the second eigenvalue has been considered by several authors([2$\sim$ 5]). Dasong Cao and Hong Yuan showed that for a simple graph $\lambda_2(G) = -1$ iff G is complete ([6]), they also established in [7] that (1) $\lambda_3(G) < -1$ iff $G \cong P_3$; (2)$\lambda_3(G) = -1$ iff G^c (the complement of G) is isomorphic to a complete bipartite plus isolated vertices; (3) there exist no graphs such that $-1 < \lambda_3(G) < -\frac{\sqrt{5}-1}{2}$. In this paper we explore the distribution of eigenvalues of a graph with $\lambda_3(G) < 0$ and obtain that:

If $\lambda_3(G) < 0$, and some $\lambda_k(G) = -1, \quad 2 \le k \le [\frac{n}{2}]$, then

$$\lambda_j(G) = -1, \quad j = k, k+1, \cdots, n-k+1.$$

The techniques and ideas are in light of matrix theory and graph theory.

2. Lemmas and Results.

Lemma 2.1([7]). *Let G be a graph with $n \ge 2$ vertices. Then for $k \ge 2$,*

$$\lambda_k(G) + \lambda_{n-k+2}(G^c) \le -1 \le \lambda_k(G) + \lambda_{n-k+1}(G^c).$$

Lemma 2.2([7]). *For every graph G with at least four vertices,*

$$\lambda_3(G) \ge -1.$$

Moreover, if G^c is not bipartite, then $\lambda_3(G) \ge 0$.

Lemma 2.3([1]). *If G is bipartite, then*

$$\lambda_i(G) = -\lambda_{n-i+1}(G),$$

for $1 \le i \le n$.

Lemma 2.4([1]). *Let G be a graph with n vertices. Then G has only one positive eigenvalue iff G is a complete multipartite graph plus isolated vertices.*

Lemma 2.5. *Let G be a graph with n vertices. Then*
(1) G is complete iff $\lambda_2(G) < 0$;
(2) G is complete iff we have

$$\lambda_1(G) = n - 1;$$

(3) G is complete iff we have

$$\lambda_j(G) = -1, \quad j = 2, 3, \cdots, n.$$

Proof. The assertions are by [6, Theorem 1] and [8,P_{37}, Theorem(2.35)].

Theorem 2.6. *Let G be a graph with $n (n \geq 4)$ vertices. If $\lambda_3(G) < 0$, and there exists an index k, $2 \leq k \leq [\frac{n}{2}]$, such that $\lambda_k(G) = -1$, then*

$$\lambda_j(G) = -1, \quad j = k, k+1, \cdots, n-k+1.$$

Furthermore, if G is unconnected and has no isolated vertices, then G is isomorphic to $G_1 \oplus G_2$, where G_i, i=1,2, are complete and therefore we have

$$\lambda_j(G) = -1, \quad j = 3, 4, \cdots, n.$$

Proof. By Lemma 2.1, we have

$$\lambda_{n-k+1}(G^c) \geq 0, \quad \lambda_{n-k+2}(G^c) \leq 0.$$

On the other hand, by Lemma 2.2, $\lambda_3(G) < 0$ implies that G^c is bipartite. Thus, using Lemma 2.3, it yields

$$0 \leq \lambda_{n-k+1}(G^c) = -\lambda_k(G^c).$$

Since

$$0 \leq \lambda_{n-k+1}(G^c) \leq \lambda_{n-k}(G^c) \leq \cdots \leq \lambda_k(G^c) \leq 0,$$

we have

$$\lambda_k(G^c) = 0.$$

Now, we use Lemma 2.1 to the graph G^c, then

$$\begin{aligned}&\lambda_k(G^c) + \lambda_{n-k+1}(G)\\&= \lambda_{n-k+1}(G)\\&\geq -1.\end{aligned}$$

Noticing that $n-k+1 \geq n - [\frac{n}{2}] + 1 \geq [\frac{n}{2}] + 1 > k$, it leads to

$$\lambda_k(G) = -1 \geq \lambda_{k+1}(G) \geq \lambda_{k+2}(G) \geq \cdots \geq \lambda_{n-k+1}(G) \geq -1.$$

i.e

$$\lambda_j(G) = -1, \quad j = k, k+1, \cdots, n-k+1.$$

Furthermore, if G is unconnected and has no isolated vertices, G has exactly two components $G_i (i = 1, 2)$ due to $\lambda_3(G) < 0$. Therefore by Lemma 2.4 and Lemma 2.5 (1) G_i are complete. This yields the following by Lemma 2.5(3)

$$\lambda_j(G) = -1, j = 3, 4, \cdots, n.$$

The proof is now complete. $\blacksquare$

Taking advantage of Theorem 2.6, we have the following results:
(I) For the case k=2(i.e. $\lambda_2(G) = -1$), by Theorem 2.6

$$\lambda_j(G) = -1, j = 2, 3, \cdots, n-1.$$

and since $\sum_{j=1}^{n} \lambda_j(G) = 0$, it gives

$$\lambda_1(G) \geq n-1.$$

But, on the other hand, we have that $\lambda_1(G) \leq n-1$. Consequently,

$$\lambda_1(G) = n-1, \lambda_j(G) = -1, \quad j = 2, 3, \cdots, n,$$

which states that G is complete. This is a known result given in [6].
(II) The case k=3(i.e.$\lambda_3(G) = -1$). In this case, we need only consider the case $\lambda_2(G) \geq 0$ due to Lemma 2.5. By Theorem 2.6, then we have

$$\lambda_j(G) = -1, \quad j = 3, 4 \cdots, n-2.$$

COROLLARY 2.7. *Let G be a graph with $n(\geq 6)$ vertices. Then $\lambda_3(G) = -1$ implies that*

$$\lambda_j(G) = -1, \quad j = 3, 4, \cdots, n-2.$$

An example: Let

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

be the adjacency matrix of a graph with eight vertices. Then by MATHEMATICA we obtain that

$$\begin{cases} \lambda_1(G) = 5.24384 \\ \lambda_2(G) = 1.60317 \\ \lambda_3(G) = -0.182062 \\ \lambda_4(G) = -0.9999 \\ \lambda_5(G) = -1 \\ \lambda_6(G) = -1 \\ \lambda_7(G) = -1.53035 \\ \lambda_8(G) = -2.1346 \end{cases}$$

On the other hand, since it is easy to verify that matrix $r(I + A(G)) = 5$ (the rank of matrix $I + A(G)$), we infer that

$$\lambda_4(G) = \lambda_5(G) = \lambda_6(G) = -1.$$

Compared with Theorem 2.6, this is the special case that k=4, n=8.

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