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SOME PROPERTIES COMPLEMENTARY TO BRUALDI-LI MATRICES

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Abstract. In this paper we derive new properties complementary to an $2n \times 2n$ Brualdi-Li tournament matrix B_{2n}. We show that B_{2n} has exactly one positive real eigenvalue and one negative real eigenvalue and, as a by-product, reprove that every Brualdi-Li matrix has distinct eigenvalues. We then bound the partial sums of the real parts and the imaginary parts of its eigenvalues. The inverse of B_{2n} is also determined. Related results obtained in previous articles are proven to be corollaries.

Keywords: tournament matrix; Brualdi-Li matrix; eigenvalue; Perron value

MSC 2010: 05C20, 15A15

1. Introduction

For a matrix or vector X, let X^* and X^t stand for the transpose conjugate and the transpose, respectively. An $n \times n$ zero-one matrix A is *tournament* if

$$A + A^t = J - I, \tag{1.1}$$

where J is the all ones matrix. A $2n \times 2n$ tournament matrix T is *almost regular* if it has n row sums equal to $n-1$ and n row sums equal to n. In [3], Brualdi and Li conjectured that for each n, the $2n \times 2n$ tournament matrix that maximizes the Perron value (the largest eigenvalue) can be written as

$$B_{2n} = \begin{pmatrix} U_n & U_n^t \\ U_n^t + I & U_n \end{pmatrix},$$

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where U_n denotes the matrix with all ones above the main diagonal, and all zeros on and below the diagonal (this matrix is often called the $n \times n$ transitive tournament). Notice that B_{2n} is an almost regular tournament matrix. To the best of our knowledge, the conjecture is still open. However, there has been great progress made on it. A strong result about this conjecture was obtained by Kirkland [13], who proved that, for sufficiently large even n, a tournament matrix T which maximizes the Perron value must be almost regular. More results about the Perron value and its asymptotic related properties can be found in [14], [15]. Motivated by the conjecture on the maximal spectral radius property of the Brualdi-Li matrix B_{2n}, we derive further properties of this matrix, which, we think, can help in considering the conjecture.

For an $n \times n$ tournament matrix A, the vector $s = A \cdot \mathbf{1}$ is called the *score vector* of A, and if $s = (n-1)/2 \cdot \mathbf{1}$, then A is said to be *regular*. The score vector can be used to obtain information about eigenvalues of A, see [16], [18]. Note that the score vector s satisfies $s^t \cdot \mathbf{1} = n(n-1)/2$ and $s^t s \geqslant n(n-1)^2/4$ with equality if and only if it is regular. These properties are useful in localizing the eigenvalues of a tournament matrix.

Tournament matrices have been well studied in the past decades (see [2], [6], [7], [9], [23] and their references). A wealth of literature focuses on deriving algebraic or combinatorial attributes of the matrices because of their interplays between matrix/graph theoretic and spectral properties [1], [2], [6], [16], [20], [23]. Originally Brauer and Gentry [1], [2] showed that if λ is an eigenvalue of a tournament matrix A of order n, then $-1/2 \leqslant \operatorname{Re}\lambda \leqslant (n-1)/2$ and $|\operatorname{Im}\lambda| \leqslant \sqrt{n(n-1)/6}$. Then Moon and Pullman extended their idea to derive similar results for the generalized tournament matrices [21]. And, subsequently, Maybee and Pullman [18] considered more general matrices, the pseudo-tournament matrices, in which they introduced h-hypertournament matrices and showed that $-1/2 \leqslant \operatorname{Re}\lambda \leqslant (n-1)/2$ for the h-hypertournament matrices.

In Section 2 we describe the preliminaries and fundamentals and then, in Section 3, we consider the distribution of the eigenvalues of the Brualdi-Li matrix B_{2n} and bound the partial sums of the real parts and imaginary parts of the eigenvalues of B_{2n}. The inverse of B_{2n} is obtained in Section 4. Several results obtained in the previous articles are proven to be special cases to the ones obtained in this paper.

For convenience, except the notations mentioned above, we will also use the following ones:

$\mathbb{C}^n(\mathbb{R}^n)$: the n-dimensional complex (real) Euclidean vector space,

$\lambda_i(A)$: the i-th eigenvalue of matrix A,

$\operatorname{Re}\lambda_i(A)$: the real part of $\lambda_i(A)$,

$\operatorname{Im}\lambda_i(A)$: the imaginary part of $\lambda_i(A)$,
$J = \mathbf{1}\cdot\mathbf{1}^t$: the all ones matrix of an appropriate size,
$\varrho(A)$: the spectral radius of matrix A (the Perron value, if A is nonnegative),
$\|\cdot\|_2$: the Euclidean norm.

In addition, for an $n\times n$ matrix A we assume that

$$\operatorname{Re}\lambda_1(A) \geqslant \operatorname{Re}\lambda_2(A) \geqslant \ldots \geqslant \operatorname{Re}\lambda_n(A),$$
$$\operatorname{Im}\lambda_{i_1}(A) \geqslant \operatorname{Im}\lambda_{i_2}(A) \geqslant \ldots \geqslant \operatorname{Im}\lambda_{i_n}(A),$$

where $\{i_1, i_2, \ldots, i_n\}$ is a permutation of $\{1, 2, \ldots, n\}$.

2. Preliminaries and lemmas

In this section we describe the fundamentals that are crucial for our consideration.

Lemma 2.1. *Let $x = (x_1, x_2, \ldots, x_n)^t$ and $y = (y_1, y_2, \ldots, y_n)^t$ be two vectors in $\mathbb{R}^n$ satisfying*

$$x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n, \quad y_1 \geqslant y_2 \geqslant \ldots \geqslant y_n.$$

Then the following four statements are equivalent:

(a) *$y = Sx$ for a doubly stochastic matrix S;*
(b) $\sum\limits_{i=1}^{k} x_i \geqslant \sum\limits_{i=1}^{k} y_i$, *$k = 1, 2, \ldots, n-1$, and* $\sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} y_i$;
(c) $\sum\limits_{i=1}^{n} \varphi(x_i) \geqslant \sum\limits_{i=1}^{n} \varphi(y_i)$, *for any continuous convex function φ;*
(d) *there exists an $n\times n$ Hermitian matrix with eigenvalues $x_1, x_2, \ldots, x_n$ and diagonal elements $y_1, y_2, \ldots, y_n$.*

P r o o f. A proof of the equivalences of statements (a), (b), and (c) is shown in [8]; the equivalence of statements (a) and (d) is proven in [10], [19]. □

The Perron value of an almost regular tournament matrix has been well studied. Let $T_{\max}$ be an almost regular tournament matrix that has the maximum spectral radius over all $2n\times 2n$ tournament matrices. Then combining the result obtained by Friedland in [5] with the one obtained by Kirkland in [12] yields the following "asymptotic" formula:

$$\varrho(T_{\max}) = \frac{2n-1}{2} - \frac{\gamma_n}{2n} + O\Big(\frac{1}{n^2}\Big),$$

where $0.375\ldots \leqslant \gamma_n \leqslant 0.380797\ldots$ The following lemma was originally obtained by Kirkland in [16] and subsequently also by Savchenko (see Acknowledgement). We will use this lemma in bounding the partial sums of the real parts of the eigenvalues of an almost regular tournament matrix later in the paper.

Lemma 2.2 ([16], Corollary 1.4). *Let T be a $2n\times 2n$ almost regular tournament matrix. Then*

$$\varrho(T)\geqslant \frac{n-1}{2}+\frac{n}{2}\sqrt{1-\frac{1}{n^2}}.$$

The following lemma is applied to bound the partial sums of the real parts and the partial sums of the imaginary parts of the eigenvalues of a tournament matrix.

Lemma 2.3. *Let A be an $n\times n$ tournament matrix. Then there exists a collection of numbers $q_1,q_2,\ldots,q_n$ satisfying $1\geqslant q_1\geqslant q_2\geqslant\ldots\geqslant q_n\geqslant 0$ and $\sum\limits_j q_j=1$ such that*

$$2\operatorname{Re}\lambda_1(A)=nq_1-1,\ 2\operatorname{Re}\lambda_2(A)=nq_2-1,\ \ldots,\ 2\operatorname{Re}\lambda_n(A)=nq_n-1,$$

which imply

$$\sum_{i=1}^{k}\operatorname{Re}\lambda_i(A)\leqslant\frac{n-k}{2},\quad k=1,2,\ldots,n-1,$$

and, in particular, $-1/2\leqslant\operatorname{Re}\lambda_n(A)$, $\operatorname{Re}\lambda_1(A)\leqslant(n-1)/2$.

P r o o f. By the well-known Bendixon's inequalities [11] we immediately have that

$$-\frac{1}{2}\leqslant\operatorname{Re}\lambda_j(A)\leqslant\frac{n-1}{2},\quad j=1,2,\ldots,n.$$

On the other hand, for all j, set $q_j=(2\operatorname{Re}\lambda_j+1)/n$. Then we have $q_j\geqslant q_{j+1}\geqslant 0$ and $\sum\limits_{j=1}^{k}q_j\leqslant 1$, for all $k<n$, and $\sum\limits_{j}^{n}q_j=1$. The claim follows trivially from these facts. $\square$

From Lemma 2.3, one sees that for an $n\times n$ tournament matrix A, the equality

$$\operatorname{Re}\lambda_k(A)+\operatorname{Re}\lambda_{k-1}(A)+\ldots+\operatorname{Re}\lambda_1(A)=\frac{n-k}{2},\quad k\geqslant 1 \tag{2.1}$$

holds if and only if A has at least $n-k$ eigenvalues with real part equal to $-1/2$. In [17], Theorem 3, Kirkland and Shader derived if and only if conditions that a tournament matrix satisfies $\varrho(A)=(n-k)/2$ and $\lambda_j(A)=0$, $j=2,3,\ldots,k$. Since their conditions satisfy equation (2.1), we immediately have that

$$\operatorname{Re}\lambda_j(A)=-\frac{1}{2},\quad j=k+1,k+2,\ldots,n.$$

From Lemma 2.3 one also sees that for an $n\times n$ tournament matrix A,

$$\lambda_1(A)\geqslant\frac{n-2}{4}+p\quad\text{implies}\quad\sum_{j=2}^{k}\operatorname{Re}\lambda_j(A)\leqslant\frac{n-2k+2}{4}-p,\quad\forall\,2\leqslant k\leqslant n, \tag{2.2}$$

where p is a real number. This property will be used to bound the (partial) sums of the real parts of the eigenvalues of A later in this paper.

3. The distributions and bounds of the eigenvalues of B_{2n}

In [9] Kirkland derived several eigen-properties of Brualdi-Li matrix B_{2n} and showed that every Brualdi-Li matrix has distinct eigenvalues and so it is diagonalizable. In this section we derive further properties of this matrix and, as a by-product, reprove that every Brualdi-Li matrix has distinct eigenvalues.

Theorem 3.1. *The Brualdi-Li matrix B_{2n} has exactly one real positive eigenvalue, one real negative eigenvalue, and its all other eigenvalues are distinct complex numbers with negative real part and satisfy*

$$-\frac{1}{2} \leqslant \operatorname{Re} \lambda_{2n}(B) \leqslant \ldots \leqslant \operatorname{Re} \lambda_2(A) \leqslant -\frac{1}{2} + \frac{1}{2n} \frac{1}{1+\sqrt{1-1/n^2}}.$$

Proof. From Lemma 2.3 we have $\operatorname{Re} \lambda_2(A) + \lambda_1(A) \leqslant (2n-2)/2$. Applying the inequality in Lemma 2.2 yields

$$\begin{aligned}
\operatorname{Re} \lambda_2(A) \leqslant & \frac{2n-2}{2} - \frac{n-1}{2} - \frac{n}{2}\sqrt{1-\frac{1}{n^2}} \\
= & -\frac{1}{2} + \frac{n}{2} - \frac{n}{2}\sqrt{1-\frac{1}{n^2}} \\
= & -\frac{1}{2} + \frac{1}{2n} \frac{1}{1+\sqrt{1-1/n^2}}.
\end{aligned}$$

Because B_{2n} is known to have distinct eigenvalues [9], it suffices to show that B_{2n} has exactly one real negative eigenvalue λ_2. From Theorem 5 of [9], the characteristic polynomial $c(\lambda)$ of B_{2n} is given by

$$c(\lambda) = \lambda^{2n} - \sum_{j=0}^{n-1} (n-1-2j)(\lambda+1)^{2(n-j-1)}\lambda^{2j}.$$

Rewriting $c(\lambda)$ we have

$$c(\lambda) = \lambda^{2n} - \sum_{j=0}^{n-1} (n-1-2j)(\lambda+1)^{2(n-j-1)}\lambda^{2j}$$

$$
\begin{aligned}
&= \lambda^{2n} - \sum_{j=0}^{\lfloor n/2\rfloor-1} (n-1-2j)(\lambda+1)^{2(n-j-1)}\lambda^{2j} \\
&\quad - \sum_{j=\lfloor n/2\rfloor}^{n-1} (n-1-2j)(\lambda+1)^{2(n-j-1)}\lambda^{2j} \\
&= \lambda^{2n} - \sum_{j=0}^{\lfloor n/2\rfloor-1} (n-1-2j)\big[(\lambda+1)^{2(n-j-1)}\lambda^{2j} - (\lambda+1)^{2j}\lambda^{2(n-j-1)}\big] \\
&= \lambda^{2n} - \sum_{j=0}^{\lfloor n/2\rfloor-1} (n-1-2j)[f_j(\lambda) - f_j(-\lambda-1)],
\end{aligned}
$$

where $f_j(\lambda) = (\lambda+1)^{2(n-j-1)}\lambda^{2j}$.

Noting that if $\lambda \leqslant -1/2 + (2n)^{-1}/(1+\sqrt{1-1/n^2})$, then

$$
(n-1)\lambda + j \leqslant -\frac{n}{2} + \frac{1}{2} + \frac{n-1}{2n}\frac{1}{1+\sqrt{1-1/n^2}} + \left\lfloor \frac{n}{2} \right\rfloor - 1 < 0,
$$

and, for all $\lambda \in (-1, -1/2 + (2n)^{-1}/(1+\sqrt{1-1/n^2})]$,

$$
\begin{aligned}
f_j'(\lambda) &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\big((\lambda+1)^{2(n-j-1)}\lambda^{2j}\big) \\
&= 2(n-j-1)(\lambda+1)^{2(n-j-1)-1}\lambda^{2j} + 2j(\lambda+1)^{2(n-j-1)}\lambda^{2j-1} \\
&= 2(\lambda+1)^{2(n-j-1)-1}\lambda^{2j-1}((n-1)\lambda+j) > 0,
\end{aligned}
$$

we see that $f_j(\lambda)$ is strictly increasing. This implies that the function $g(\lambda) = f_j(\lambda) - f_j(-\lambda-1)$ is also strictly increasing, and so is

$$
\sum_{j=0}^{\lfloor n/2\rfloor-1} (n-1-2j)[f_j(\lambda) - f_j(-\lambda-1)].
$$

By this property and noting that B_{2n} has exactly one positive eigenvalue, $c(\lambda) = 0$ has exactly one real negative root. Therefore, combining this result with Theorem 3 of [9] we conclude that B_{2n} has exactly one real positive eigenvalue and one real negative eigenvalue, and its all other eigenvalues are distinct complex numbers with negative real part. $\square$

Theorem 3.2. *Let $\lambda \neq 0$ and $(\lambda+1)/\lambda = \mathrm{e}^z$. Then λ is an eigenvalue of the Brualdi-Li matrix B_{2n} if and only if z is a solution of the following hyperbolic equation:*

$$\tanh nz \cdot \tanh \frac{z}{2} + \coth \frac{z}{2} = 2n. \tag{3.1}$$

P r o o f. From the proof of Theorem 3.1, the characteristic polynomial $c(\lambda)$ of B_{2n} can be rewritten as

$$\begin{aligned}
c(\lambda) &= \lambda^{2n} - \sum_{j=0}^{\lfloor n/2\rfloor-1} (n-1-2j)\big[(\lambda+1)^{2(n-j-1)}\lambda^{2j} - (\lambda+1)^{2j}\lambda^{2(n-j-1)}\big] \\
&= \lambda^{2n}\Bigg(1 - \frac{1}{\lambda^2}\Big(\frac{\lambda+1}{\lambda}\Big)^{n-1} \sum_{j=0}^{\lfloor n/2\rfloor-1} (n-2j-1) \\
&\quad \times \Bigg[\Big(\frac{\lambda+1}{\lambda}\Big)^{n-2j-1} - \Big(\frac{\lambda}{\lambda+1}\Big)^{n-2j-1}\Bigg]\Bigg).
\end{aligned}$$

Now, for any nonzero λ (complex or real), define $(\lambda+1)/\lambda = \mathrm{e}^z$. Then

$$\begin{aligned}
\sum_{j=0}^{\lfloor n/2\rfloor-1} (n-2j-1)\Bigg(\Big(\frac{\lambda+1}{\lambda}\Big)^{n-2j-1} - \Big(\frac{\lambda}{\lambda+1}\Big)^{n-2j-1}\Bigg) \\
&= \sum_{j=0}^{\lfloor n/2\rfloor-1} \frac{\mathrm{d}}{\mathrm{d}z}\big(\mathrm{e}^{(n-2j-1)z} + \mathrm{e}^{-(n-2j-1)z}\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\sum_{j=0}^{\lfloor n/2\rfloor-1} \big(\mathrm{e}^{(n-2j-1)z} + \mathrm{e}^{-(n-2j-1)z}\big) \\
&= 2\frac{\mathrm{d}}{\mathrm{d}z}\sum_{j=0}^{\lfloor n/2\rfloor-1} \cosh(n-2j-1)z.
\end{aligned}$$

It is easily checked that, similar to the formula in [22], page 73, we have

$$\sum_{j=0}^{\lfloor n/2\rfloor-1} \cosh(n-2j-1)z = \begin{cases} \dfrac{\sinh nz}{2\sinh z}, & n \text{ even}, \\[2ex] \dfrac{\sinh nz}{2\sinh z} - \dfrac{1}{2}, & n \text{ odd}, \end{cases}$$

differentiating both sides of the above identity and then proceeding a simple manipulation by setting $c(\lambda) = 0$ yield

$$2\frac{\mathrm{d}}{\mathrm{d}z}\sum_{j=0}^{\lfloor n/2\rfloor-1} \cosh(n-2j-1)z = \frac{n\cosh nz\sinh z - \sinh nz\cosh z}{\sinh^2 z} = \frac{\mathrm{e}^{-nz}}{2(\cosh z-1)}.$$

Since $\mathrm{e}^{-nz} = \cosh nz - \sinh nz$, we have

$$\begin{aligned}
\frac{n\cosh nz \sinh z - \sinh nz \cosh z}{\sinh^2 z} &= \frac{-\sinh nz + \cosh nz}{2(\cosh z - 1)} \\
\iff \frac{n\cosh nz}{\sinh z} - \frac{\cosh nz}{2(\cosh z - 1)} &= \frac{\sinh nz \cosh z}{\sinh^2 z} - \frac{\sinh nz}{2(\cosh z - 1)} \\
&= \frac{\sinh nz}{\sinh^2 z}\left(\cosh z - \frac{\sinh^2 z}{2(\cosh z - 1)}\right) \\
&= \frac{\sinh nz}{2\sinh^2 z}(\cosh z - 1)
\end{aligned}$$

(applying the formulas $\sinh z = 2\sinh(z/2)\cosh(z/2)$, $\cosh z - 1 = 2\sinh^2(z/2)$)

$$\begin{aligned}
&\iff \left(2n - \frac{\sinh z}{\cosh z - 1}\right)\cosh nz = \frac{\sinh nz}{\cosh \frac{z}{2}} \sinh \frac{z}{2} \\
&\iff \tanh nz \cdot \tanh \frac{z}{2} = 2n - \coth \frac{z}{2}.
\end{aligned}$$

Noting that if λ is the real negative eigenvalue of B_{2n} from Theorem 3.1, we have $\lambda > -1/2$, and so, letting $z = -x + \mathrm{i}\pi$ $(x > 0)$ implies that $(\lambda + 1)/\lambda = -\mathrm{e}^{-x}$, equivalently,

$$\lambda = \frac{1}{-\mathrm{e}^{-x} - 1}.$$

If λ is the real positive eigenvalue (the Perron value) of B_{2n}, then we may let $z = x > 0$ so that $(\lambda + 1)/\lambda = \mathrm{e}^{x}$, which implies

$$\lambda = \frac{1}{\mathrm{e}^{x} - 1}.$$

The prove is thus completed. $\square$

Remark 3.1. One can easily show that equation (3.1) has $2n$ distinct roots and among them there are exactly one real positive root and one real negative real root. Therefore, Theorem 3.2 implies that B_{2n} has $2n$ distinct eigenvalues, generating an alternative proof of Theorem 3 of [9].

In the following we derive the upper bounds on the partial sums of the real parts and the imaginary parts of the eigenvalues of B_{2n}. Since B_{2n} has exactly $2n - 2$ eigenvalues with nonzero imaginary parts, we will only take care of the $n-1$ positive imaginary parts of the complex eigenvalues.

Theorem 3.3. *The partial sums of the real parts of the eigenvalues of Brualdi-Li matrix B_{2n} satisfy*

$$-\frac{k-1}{2} \leqslant \sum_{j=2}^{k} \operatorname{Re}\lambda_j(A) \leqslant -\frac{k-1}{2} + \frac{1}{2n}\frac{1}{1+\sqrt{1-1/n^2}}, \quad \forall k,\ 2 \leqslant k \leqslant 2n,$$

while the partial sums of the imaginary parts of the complex eigenvalues of Brualdi-Li matrix B_{2n} satisfy

$$\sum_{j=1}^{k} \operatorname{Im}\lambda_{i_j}(B_{2n}) \leqslant \frac{1}{2}\sum_{j=1}^{k} \cot\frac{(2j-1)\pi}{4n}, \quad \forall k,\ 1 \leqslant k \leqslant n-1,$$

where $\operatorname{Im}\lambda_{i_j}(B_{2n}) \geqslant \operatorname{Im}\lambda_{i_{j+1}}(B_{2n}) > 0$, *for all* $1 \leqslant j \leqslant n-1$.

Proof. By combining Theorem 3.1 and Lemmas 2.2 and 2.3, it is easily checked that the partial sums of the real parts of the eigenvalues of B_{2n} satisfy the inequalities asserted. Below we consider the imaginary parts. By the Schur decomposition theorem [11] there exists a unitary matrix Q such that

$$Q^*AQ = \begin{pmatrix} \lambda_1(A) & & a_{ij} \\ & \ddots & \\ 0 & & \lambda_n(A) \end{pmatrix},$$

which yields

$$Q^*(B_{2n} - B_{2n}^t)Q = \begin{pmatrix} 2\mathrm{i}\ \operatorname{Im}\lambda_1(B_{2n}) & & a_{ij} \\ & \ddots & \\ -a_{ij}^* & & 2\mathrm{i}\ \operatorname{Im}\lambda_{2n}(B_{2n}) \end{pmatrix}.$$

Let $P_n = (B_{2n} - B_{2n}^t)/(2\mathrm{i})$. Then P_n is an Hermitian matrix, and so by Lemma 2.1 we have

$$\sum_{j=1}^{k} \operatorname{Im}\lambda_{i_j}(B_{2n}) \leqslant \sum_{j=1}^{k} \lambda_j(P_n), \quad k = 1, 2, \ldots, 2n-1, \tag{3.2}$$

$$\sum_{j=1}^{2n} \operatorname{Im}\lambda_j(B_{2n}) = \sum_{j=1}^{2n} \lambda_j(P_n) = 0.$$

We now determine the eigenvalues of P_n and consider the eigenvalues of the following "shifted" matrix:

$$I + 2\mathrm{i}P_n = I + B_{2n} - B_{2n}^t = \begin{pmatrix} I + U_n - U_n^t & U_n^t - U_n - I \\ U_n^t - U_n + I & I + U_n - U_n^t \end{pmatrix} = \begin{pmatrix} V_n & -V_n \\ V_n^t & V_n \end{pmatrix},$$

where $V_n = I + U_n - U_n$, for example, if $n = 4$, then

$$I + 2\mathrm{i}P_4 = \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\ \hline 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \end{array}\right) = \begin{pmatrix} V_2 & -V_2 \\ V_2^t & V_2 \end{pmatrix}.$$

Noting that $V_n = I + \eta + \dots \eta^{n-1}$, where

$$\eta = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ & \dots & & \\ -1 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{n-1} \\ -1 & 0 \end{pmatrix},$$

from [4], Probl. (4) on p. 84, $\eta^n = -I$ and η has eigenvalues $\sigma_j = \cos((2j-1)\pi/n) + \mathrm{i}\sin((2j-1)\pi/n)$, $j = 1, 2, \dots, n$. This implies that the eigenvalues of V_n are $\lambda'_j = 2/(1-\sigma_j)$. On the other hand, it is easily checked that V_n is a normal matrix. So we have

$$\det(\lambda I - (I + 2\mathrm{i}P_n)) = \det((\lambda I - V_n)^2 + V_n^t V_n) = \prod_{j=1}^{n} \{(\lambda - \lambda'_j)^2 + |\lambda'_j|^2\},$$

and therefore, $\lambda_j(I + 2\mathrm{i}P) = \lambda'_j \pm \mathrm{i}|\lambda'_j|$, $j = 1, 2, \dots, n$. Through a simple calculation we obtain that $\lambda'_j = 1 + \mathrm{i}\cot((2j-1)\pi/(2n))$ and $|\lambda'_j| = \sqrt{1 + \cot^2((2j-1)\pi/(2n))}$, and so plugging these values into the equations yields

$$\lambda_j(I + 2\mathrm{i}P_n) = 1 + \mathrm{i}\left(\cot\frac{(2j-1)\pi}{2n} \pm \sqrt{1 + \cot^2\frac{(2j-1)\pi}{2n}}\right), \quad j = 1, 2, \dots, n.$$

Equivalently,

$$\begin{aligned} \lambda_j(P_n) &= \frac{1}{2}\left(\cot\frac{(2j-1)\pi}{2n} \pm \sqrt{1 + \cot^2\frac{(2j-1)\pi}{2n}}\right) \\ &= \begin{cases} \dfrac{1}{2}\cot\dfrac{(2j-1)\pi}{4n}, & j \leqslant n, \\ -\dfrac{1}{2}\tan\dfrac{(2j-1)\pi}{4n}, & j > n. \end{cases} \end{aligned}$$

Plugging these expressions into (3.2) results in the proof of the theorem. □

Notice that in Theorem 3.3, when $k = 1$, the upper bound coincides with Pick's bound derived in [7], that is, $\operatorname{Im} \lambda_{i_1}(B_{2n}) \leqslant \frac{1}{2}\cot(\pi/(4n))$.

Corollary 3.1. *Let $\lambda_1(B_{2n})$ and $\lambda_2(B_{2n})$ be the positive and negative eigenvalues of B_{2n}, respectively. Then the complex eigenvalues $\lambda_j(B_{2n})$, $j \geqslant 3$, satisfy*

$$2\Big(1 - \frac{1}{2n-1}\Big) < \prod_{j=3}^{2n} \lambda_j(B_{2n}) < 2\Big(1 + \frac{1}{n-1}\Big),$$

which implies

$$\lim_{n\to\infty} \prod_{j=1}^{n-1} \Big(\frac{1}{4} + (\operatorname{Im} \lambda_{i_j}(B_{2n}))^2\Big) = \lim_{n\to\infty} \prod_{j=3}^{2n} \lambda_j(B_{2n}) = 2.$$

P r o o f. From the characteristic polynomial $c(\lambda)$ of B_{2n} (see the proof of Theorem 3.1), the determinant of B_{2n} satisfies

$$\det(B_{2n}) = -\frac{2n-2}{2}.$$

From Lemma 2.2 and Theorem 3.1 we have

$$\frac{2n-1}{2} > \lambda_1(B_{2n}) > \frac{2n-2}{2}, \quad \frac{1}{2} > |\lambda_2(B_{2n})| > \frac{1}{2}\Big(1 - \frac{1}{n}\Big) = \frac{n-1}{2n}.$$

Combining these properties,

$$\prod_{j=3}^{2n} \lambda_j(B_{2n}) = -\frac{2n-2}{2}\,\frac{1}{\lambda_1(B_{2n})\lambda_2(B_{2n})} < \frac{1}{|\lambda_2(B_{2n})|} < \frac{2n}{n-1} = 2\Big(1 + \frac{1}{n-1}\Big).$$

On the other hand,

$$\prod_{j=3}^{2n} \lambda_j(B_{2n}) = -\frac{2n-2}{2}\,\frac{1}{\lambda_1(B_{2n})\lambda_2(B_{2n})} > \frac{2n-2}{2}\,\frac{4}{2n-1} = 2\Big(1 - \frac{1}{2n-1}\Big),$$

which implies $\lim\limits_{n\to\infty} \prod\limits_{j=3}^{2n} \lambda_j(B_{2n}) = 2$. The rest of proof follows from Theorem 3.1. $\square$

4. The inverse of Brualdi-Li matrix B_{2n}

In this section we derive the inverse of Brualdi-Li matrix B_{2n}. For the sake of convenience, we introduce some conventions. Let

$$\xi = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ & \dots & & \\ 1 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{2n-2} \\ 1 & 0 \end{pmatrix}$$

be the $(2n-1) \times (2n-1)$ basic circulant matrix and let $C = I + \xi + \ldots + \xi^{n-2}$. Then

$$B_{2n} = \begin{pmatrix} a & C \\ 1 & b^t \end{pmatrix},$$

where a is the $(2n-1)$-dimensional zero-one column vector whose first n entries are 0 and the last $n-1$ entries are 1, and b is the $(2n-1)$-dimensional zero-one column vector whose first $n-1$ entries are 1 and the last n entries are 0. For example, if $n = 4$, we have

$$B_8 = \left(\begin{array}{c|ccccccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right).$$

We now solve the following equation for x, y, T and z.

$$\begin{pmatrix} a & C \\ 1 & b^t \end{pmatrix} \begin{pmatrix} x^t & y \\ T & z \end{pmatrix} = I_{2n},$$

where T is an $(2n-1) \times (2n-1)$ matrix and y is a number. Comparing both sides we get

$$ax^t + CT = I_{2n-1}, \quad ay + Cz = 0, \quad x^t + b^t T = 0, \quad y = 1 - b^t z.$$

Combining the second and fourth equations, we have $a(1 - b^t z) + Cz = 0 \implies a = (ab^t - C)z$, and $z = (0, \ldots, 0, -1)^t$, and so $y = 1 - b^t z = 1$. Observing that $B_{2n} = (b_{i,j})$ satisfies $b_{i,j} = b_{n+1-j,n+1-i}$, its inverse also has this property. This implies that we must have $x^t = -b^t T = (-1, 0, \ldots, 0)$. So from the equation

$ax^t + CT = I_{2n-1}$ we have $T = (C - ab^t)^{-1}$. Consequently, the inverse of B_{2n} is given by

$$B_{2n}^{-1} = \left(\begin{array}{c|c} x^t & 1 \\ \hline (C - ab^t)^{-1} & z \end{array}\right) = (b'_{i,j}), \tag{4.1}$$

where $x^t = (-1, 0, \ldots, 0)$, $z = (0, \ldots, 0, -1)^t$, and

$$ab^t = \left(\begin{array}{c|c} 0_{n,n-1} & 0_{n,n} \\ \hline J_{n-1,n-1} & 0_{n-1,n} \end{array}\right).$$

Making use of the property $b'_{i,j} = b'_{n+1-j,n+1,i}$ and performing simple calculations on $(C - ab^t)^{-1}$ in (4.1), we obtain the inverse

$$B_{2n}^{-1} = \frac{1}{n-1}\left(\begin{array}{cccccccccc|c}
1-n & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & n-1 \\ \hline
1 & 2-n & 1 & \ldots & 1 & 2-n & 1 & \ldots & 1 & 1 & 0 \\
1 & 1 & 2-n & 1 & \ldots & 1 & 2-n & 1 & \ldots & 1 & 0 \\
\vdots & \vdots & & \ddots & \ddots & & \ddots & \ddots & & \vdots & \vdots \\
1 & 1 & \ldots & & 2-n & 1 & \ldots & & 1 & 2-n & 0 \\
2-n & 1 & 1 & \ldots & 1 & 2-n & 1 & \ldots & & 1 & 0 \\
1 & 2-n & 1 & \ldots & & 1 & 2-n & 1 & \ldots & 1 & 0 \\
\vdots & \vdots & \ddots & \ddots & & & \ddots & \ddots & & \vdots & \vdots \\
1 & 1 & \ldots & 2-n & 1 & 1 & \ldots & & 1 & 2-n & 0 \\
1 & 1 & \ldots & 1 & 2-n & 1 & & \ldots & & 1 & 1-n
\end{array}\right), \tag{4.2}$$

where, in the left-bottom block, the elements $(2-n)$ are along three lines parallel to the main diagonal, the first starting from the position $(1, 2)$, the second from $(1, n+1)$, and the last from $(n, 1)$. For example, if $n = 4$, then

$$B_8^{-1} = \frac{1}{3}\left(\begin{array}{ccccccc|c}
-3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ \hline
1 & -2 & 1 & 1 & -2 & 1 & 1 & 0 \\
1 & 1 & -2 & 1 & 1 & -2 & 1 & 0 \\
1 & 1 & 1 & -2 & 1 & 1 & -2 & 0 \\
-2 & 1 & 1 & 1 & -2 & 1 & 1 & 0 \\
1 & -2 & 1 & 1 & 1 & -2 & 1 & 0 \\
1 & 1 & -2 & 1 & 1 & 1 & -2 & 0 \\
1 & 1 & 1 & -2 & 1 & 1 & 1 & -3
\end{array}\right).$$

Theorem 4.1.

(i) *The inverse of the Brualdi-Li matrix B_{2n} is given by (4.2).*

(ii) *Let σ_1 and σ_{2n} be the largest and the least singular values of B_{2n}, respectively. Then*

$$n - 1 < \sigma_1 < n, \qquad \frac{1}{4 - \dfrac{1}{n-1}} < \sigma_{2n} < \frac{1}{\sqrt{2}}.$$

P r o o f. From Lemma 2.2, we have that $(2n-2)/2 < \lambda_1(B_{2n}) \leqslant \sigma_1$. On the other hand, since $\sigma_1^2 = ||B_{2n}||_2 \leqslant ||B_{2n}||_1||B_{2n}||_\infty = n^2$, we have $n-1 < \sigma_1 < n$.

We now turn to the second part. From the last row and the first column of matrix (4.2) we have

$$\sigma_{2n}^{-1} \leqslant (\|B_{2n}^{-1}\|_\infty \|B_{2n}^{-1}\|_1)^{1/2} = \frac{4n-5}{n-1} = 4 - \frac{1}{n-1}.$$

On the other hand,

$$\sigma_{2n}^{-2} > (B_{2n}^{-1}B_{2n}^{-t})_{11} = 2.$$

Combining the two bounds yields the assertion. □

5. Concluding remarks

In this paper we derived further properties of the Brualdi-Li tournament matrices B_{2n}. We considered the distribution of their eigenvalues and derived bounds on the partial sums of the real parts and imaginary parts of the eigenvalues of B_{2n}. The inverse of B_{2n} is also obtained. Several results obtained in previous articles are proven to be special cases to the ones provided in this paper.

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