Discrete Mathematics 309 (2009) 3364–3369

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The lollipop graph is determined by its Q-spectrum[☆]

Yuanping Zhang[a,*], Xiaogang Liu[b], Bingyan Zhang[a], Xuerong Yong[c]

[a] School of Computer and Communication, Lanzhou University of Technology, Lanzhou, 730050, Gansu, PR China
[b] School of Science, Lanzhou University of Technology, Lanzhou 730050, Gansu, PR China
[c] Department of Mathematics, University of Puerto Rico at Mayaguez, P.O.Box 9018, PR 00681, USA

ARTICLE INFO

Article history:
Received 31 March 2008
Received in revised form 25 September 2008
Accepted 30 September 2008
Available online 13 November 2008

Keywords:
Spectrum of a graph
Cospectral graphs
Eigenvalues

ABSTRACT

A graph G is said to be *determined by its Q-spectrum* if with respect to the signless Laplacian matrix Q, any graph having the same spectrum as G is isomorphic to G. The *lollipop graph*, denoted by $H_{n,p}$, is obtained by appending a cycle C_p to a pendant vertex of a path P_{n-p}. In this paper, it is proved that all lollipop graphs are determined by their Q-spectra.

1. Introduction

Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)$. All graphs considered here are simple and undirected. Let matrix $A(G)$ be the *(0,1)-adjacency matrix* of G and d_k the degree of the vertex v_k. The matrix $L(G)=D(G)-A(G)$ (or $Q(G)=D(G)+A(G)$) is called the *Laplacian matrix* (or *signless Laplacian matrix*) of G, where $D(G)$ is the $n\times n$ diagonal matrix with $\{d_1,d_2,\ldots,d_n\}$ as diagonal entries. The polynomial $P_{A(G)}(\lambda)=\det(\lambda I-A(G))$ (or $P_{L(G)}(\mu)=\det(\mu I-L(G))$; $P_{Q(G)}(\nu)=\det(\nu I-Q(G))$), where I is the identity matrix, is defined as the *adjacency characteristic polynomial* (or *Laplacian characteristic polynomial*; *signless Laplacian characteristic polynomial*) of the graph G. Assume that $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$, $\mu_1\geq\mu_2\geq\cdots\geq\mu_n$ and $\nu_1\geq\nu_2\geq\cdots\geq\nu_n$ are *the adjacency eigenvalues*, *Laplacian eigenvalues* and the *signless Laplacian eigenvalues* of graph G, respectively. The *adjacency spectrum* (or *Laplacian spectrum*; *signless Laplacian spectrum*) of the graph G consists of the adjacency eigenvalues (or Laplacian eigenvalues; signless Laplacian eigenvalues). In the following, the signless Laplacian matrix, the signless Laplacian characteristic polynomial, the signless Laplacian eigenvalues and the signless Laplacian spectrum are abbreviated to *Q-matrix*, *Q-polynomial*, *Q-eigenvalues* and *Q-spectrum*, respectively.

Graphs with the same spectrum of an associated matrix M are called *cospectral* graphs with respect to M. A graph H cospectral with a graph G, but not isomorphic to G, is called a *cospectral mate* of G. Two graphs are said to be *Q-cospectral* if they have the same Q-polynomial. In [10,17], the graphs $K_{1,3}$ and $K_3\cup K_1$, the smallest Q-cospectral mates are given. Other Q-cospectral mates are shown in Fig. 1, which implies that the T-shape trees (see [18,19]) are not determined by their Q-spectra.

Which graphs are determined by their spectra seems to be a difficult problem in the theory of graph spectra. Up to now, only a few graphs with very special structures have been proved to be determined by their spectra (adjacency spectra or

☆ Supported by Foundation for Research Development of Lanzhou University of Technology and NSF of Gansu Province (3ZS051-A25-037).
* Corresponding author.
E-mail addresses: ypzhang@lut.cn (Y. Zhang), lxg.666@163.com (X. Liu), xryong@math.uprm.edu (X. Yong).

0012-365X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2008.09.052

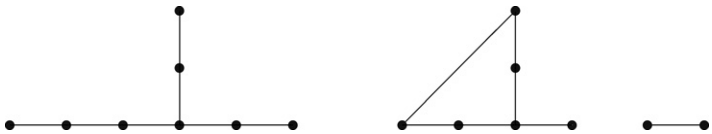

Fig. 1. The Q-cospectral graphs G and G'.

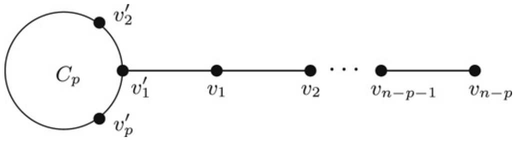

Fig. 2. The lollipop graph $H_{n,p}$.

Laplacian spectra) [4,6,9,11,13,15,17–19]. However, less graphs have been proved to be determined by their Q-spectra. The question "Which graphs are determined by their Q-spectra?" is proposed in [17]: Which linear combination of D, A, and J (the all-ones matrix) gives the most graphs determined by the spectrum? We still do not know the answer. However, there is some evidence that the Q-matrix is a good candidate, see Table 1 in [17].

For the Laplacian spectrum, it is known that the multiplicity of the eigenvalue 0 is equal to the number of components [17]. For the Q-spectrum the multiplicity of 0 gives the number of bipartite components [3]. In fact, if graph G is bipartite, its Q-spectrum equals its Laplacian spectrum (see Lemma 2.3).

In [17], the path with n vertices P_n and the disjoint union of k disjoint paths $P_{n_1} + P_{n_2} + \cdots + P_{n_k}$ are proved to be determined by their Q-spectra, respectively. Since such graphs are bipartite, the first step, it is proved that graph G' which is Q-cospectral with P_n (or $P_{n_1} + P_{n_2} + \cdots + P_{n_k}$) is also bipartite. Then, their Q-spectra and Laplacian spectra are the same (see Lemma 2.3). Finally, it is proved that G' and P_n (or $P_{n_1} + P_{n_2} + \cdots + P_{n_k}$) are isomorphic.

Also, in [17], the complete graph K_n, the regular complete bipartite graph $K_{m,m}$, the cycle C_n and the disjoint union of k disjoint cycles $C_{m_1} + C_{m_2} + \cdots + C_{m_k}$ are proved to be determined by their Q-spectra, respectively. Such graphs are all regular. If a regular graph G is determined by its spectrum with respect to one of the matrices $A(G)$; $A(\overline{G})$ ($\overline{G}$ is the complement of graph G); $L(G)$ or $Q(G)$, it is determined by its spectrum with respect to any one of the other matrices [17]. It is proved that all the above graphs are determined by their adjacency spectra, respectively. So, the graphs are also determined by their Q-spectra.

In this paper, we consider the above problem for the *lollipop graph*, denoted by $H_{n,p}$ (shown in Fig. 2), which is obtained by appending a cycle C_p to a pendant vertex of a path P_{n-p} (see [5]). In [9], the lollipop graph $H_{n,p}$ for p odd is proved to be determined by its adjacency spectrum, and all the lollipop graphs are proved to be determined by their Laplacian spectra. Also the lollipop graphs with an even cycle are proved to be determined by their adjacency spectra (Tayfeh-Rezaie [private communication] did the lollipop graphs with a cycle of length at least 6, and Boulet and Jouve [1] did the general case). In the following, it will be proved that the lollipop graphs are determined by their Q-spectra. Since the lollipop graph $H_{n,p}$ is bipartite when p is even and not when p is odd, and it is not regular, the methods used in [17] can not be helpful. In the proof, the degree sequence of graph G which is Q-cospectral to $H_{n,p}$ is determined first. Since the connectivity of G can not be obtained from its Q-spectrum, discussions are done when we suppose G is connected or not. In conclusion, G is proved to be isomorphic to $H_{n,p}$.

For the Q-spectrum of graph G and the adjacency spectrum of its line graph $\mathcal{L}(G)$, the following relations are well known (see, for example, [3]):

$$P_{A(\mathcal{L}(G))}(\lambda) = (\lambda + 2)^{m-n} P_{Q(G)}(\lambda + 2). \tag{1.1}$$

From the relation (1.1), it is known that if two graphs are Q-cospectral, then their line graphs are cospectral with respect to the adjacency matrix (Lemma 2.10). Since the lollipop graph is proved to be determined by its Q-spectrum, another question "Are line graphs of lollipop graphs determined by their adjacency spectra?" should also be answered. Unfortunately, the answer is negative. Fig. 3 gives a counter example (see Section 3 for details).

2. Preliminaries

Some valuable established results about the spectrum are summarized in this section. They will play an important role throughout this paper.

Lemma 2.1 ([8]). *Let A, B be two irreducible nonnegative $n \times n$ matrices. Then $\rho(A + B) \geq \rho(A)$, and equality holds if and only if $B = 0$, where $\rho(A)$ is the largest eigenvalue of A.*

Since the Q-matrix of a simple and connected graph G and the Q-matrix of any spanning subgraph of G are both irreducible nonnegative matrices, Lemma 2.1 implies the following corollary.

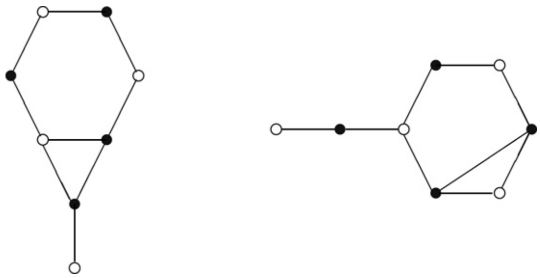

Fig. 3. The cospectral graphs $\mathcal{L}(H_{8,6})$ and G'.

Corollary 2.2. *Let G be a simple and connected graph with n vertices. If G' is a spanning subgraph of G, then $\nu_1(G') \le \nu_1(G)$, and equality holds if and only if G' is isomorphic to G.*

Lemma 2.3 ([3]). *For any bipartite graph the Q-polynomial is equal to the characteristic polynomial of the Laplacian.*

Lemma 2.4 ([12]). *If G has at least one edge, then $\mu_1(G) \ge \Delta(G) + 1$. For a connected graph G on $n > 1$ vertices, equality holds if and only if $\Delta(G) = n - 1$, where $\Delta(G)$ denotes the maximum vertex degree of G.*

Lemma 2.5. *Let G be a simple and connected graph with $n > 1$ vertices, then $\nu_1(G) \ge \Delta(G) + 1$, equality holds if and only if G is a star with n vertices.*

Proof. Let T be a spanning tree of G with $\Delta(G) = \Delta(T)$. By Corollary 2.2, $\nu_1(G) \ge \nu_1(T)$. Lemmas 2.3 and 2.4 imply that $\nu_1(T) = \mu_1(T) \ge \Delta(G) + 1$. Then $\nu_1(G) \ge \Delta(G) + 1$.

If G is a star with n vertices, i.e., $\Delta(G) = n - 1$, by Lemmas 2.3 and 2.4, $\nu_1(G) = \mu_1(K_{1,n-1}) = n = \Delta(G) + 1$.

If $\nu_1(G) = \Delta(G) + 1$. Let T' be a spanning tree of G with $\Delta(T') = \Delta(G)$. If T' is not isomorphic to G, by Corollary 2.2 and Lemmas 2.3 and 2.4, $\nu_1(G) > \nu_1(T') = \mu_1(T') \ge \Delta(G) + 1$, a contradiction. Therefore, G is a tree. Lemmas 2.3 and 2.4 imply that $\Delta(G) = n - 1$. Then G is a star with n vertices. □

Lemma 2.6 ([3]). *Let G be a graph on n vertices with vertex degrees $d_1, d_2, \ldots, d_n$ and largest Q-eigenvalue ν_1. Then*

$$2\min\{d_1, d_2, \ldots, d_n\} \le \nu_1 \le 2\max\{d_1, d_2, \ldots, d_n\}.$$

For a connected graph G, equality holds in either of these inequalities if and only if G is regular.

Lemma 2.7 ([14]). *Let the graph G have at least 2 vertices. Then*

$$\nu_1(G') \ge \nu_1(G) \ge \nu_2(G') \ge \nu_2(G) \ge \cdots \ge \nu_n(G') \ge \nu_n(G)$$

where $G' = G + e$ is the graph obtained from G by inserting a new edge e into G, n is the order of G.

Lemma 2.8. *$\nu_2(H_{n,p}) < 4$ for any lollipop graph $H_{n,p}$.*

Proof. If we delete the edge $v_1'v_2'$ from $H_{n,p}$ (see Fig. 2), we obtain an induced subgraph P_n. Since the Laplacian spectrum of the path P_n is $2 + 2\cos\frac{i\pi}{n}$ $(i = 1, 2, \ldots, n)$, Lemma 2.3 implies that the Q-spectrum of path P_n is also $2 + 2\cos\frac{i\pi}{n}$ $(i = 1, 2, \ldots, m)$. By Lemma 2.7, $\nu_2(H_{n,p}) \le \nu_1(P_n) < 4$. □

Lemma 2.9 ([3,16]). *Let G be a graph with n vertices, m edges, t triangles and vertex degrees $d_1, d_2, \ldots, d_n$. Let $T_k = \sum_{i=1}^{n} \nu_i^k$, $(k = 0, 1, \ldots)$ be the kth spectral moment for the Q-spectrum. Then*

$$T_0 = n, \qquad T_1 = \sum_{i=1}^{n} d_i = 2m, \qquad T_2 = 2m + \sum_{i=1}^{n} d_i^2, \qquad T_3 = 6t + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

Lemma 2.10 ([3]). *If two graphs are Q-cospectral, then their line graphs are cospectral with respect to the adjacency matrix.*

Lemma 2.11 ([2]). *Let C_n and P_n denote the cycle and the path on n vertices, respectively. Then*

$$P_{A(C_n)}(\lambda) = \prod_{j=1}^{n}\left(\lambda - 2\cos\frac{2\pi j}{n}\right) = 2\cos\left(n \arccos\frac{\lambda}{2}\right) - 2,$$

$$P_{A(P_n)}(\lambda) = \prod_{j=1}^{n}\left(\lambda - 2\cos\frac{\pi j}{n+1}\right) = \frac{\sin\left((n+1)\arccos\frac{\lambda}{2}\right)}{\sin\left(\arccos\frac{\lambda}{2}\right)}.$$

Let $\lambda = 2\cos\theta$, set $t^{1/2} = e^{i\theta}$, then it is useful to write the adjacency characteristic polynomial of C_n and P_n in the following form:

$$P_{A(C_n)}(t^{1/2} + t^{-1/2}) = t^{n/2} + t^{-n/2} - 2, \tag{2.1}$$

$$P_{A(P_n)}(t^{1/2} + t^{-1/2}) = t^{-n/2}(t^{n+1} - 1)/(t - 1). \tag{2.2}$$

Lemma 2.12 ([2]). *Let u be a vertex of G, $N(u)$ be the set of all vertices adjacent to u and $C(u)$ be the set of all cycles containing u. The characteristic polynomial of G satisfies*

$$P_{A(G)}(\lambda) = \lambda P_{A(G-u)}(\lambda) - \sum_{v\in N(u)} P_{A(G-u-v)}(\lambda) - 2\sum_{Z\in C(u)} P_{A(G\setminus V(Z))}(\lambda).$$

3. Main results

First, we will prove that the two graphs in Fig. 3 and their complements are cospectral with respect to the adjacency matrix, respectively.

Theorem 3.1. *The graph $\mathcal{L}(H_{8,6})$ and the graph G' given in Fig. 3 are cospectral with respect to the adjacency matrix and the same is true for their complements.*

Proof. Consider the four white vertices $\mathcal{L}(H_{8,6})$ in Fig. 3. For each white vertex v, delete the edges between v and the black neighbors, and insert edges between v and the other black vertices. It is easily checked that this operation transforms $\mathcal{L}(H_{8,6})$ into G'. Godsil and McKay (see [7], this operation is called Godsil–McKay switching) have shown that this operation leaves the adjacency spectrum of the graph and its complement unchanged. □

It is clear that $\mathcal{L}(H_{8,6})$ and G' are non-isomorphic. So $\mathcal{L}(H_{8,6})$ is not determined by its adjacency spectrum. Since also the complements of $\mathcal{L}(H_{8,6})$ and G' are cospectral with respect to the adjacency matrix, it also follows that $\mathcal{L}(H_{8,6})$ is not determined by the spectra of all its generalized adjacency matrices (see [7]).

Corollary 3.2. *Graph $\mathcal{L}(H_{8,6})$ is not determined by the spectra of all its generalized adjacency matrices.*

For the line graph of lollipop graph $H_{n,p}$, there exist a counterexample $H_{8,6}$, its line graph $\mathcal{L}(H_{8,6})$ is not determined by its adjacency spectrum. So, a graph is determined by its Q-spectrum, its line graph may be not determined by its adjacency spectrum. For all line graphs of lollipop graphs, we have the following theorem.

Theorem 3.3. *For no two non-isomorphic lollipop graphs, their corresponding line graphs have the same adjacency spectrum.*

Proof. Let $\mathcal{L}(G)$ be the line graph of graph G (recall, vertices of $\mathcal{L}(G)$ are in one-to-one correspondence with edges of G, and two vertices in $\mathcal{L}(G)$ are adjacent if and only if the corresponding edges of G are adjacent). In view of that, if two graphs are cospectral, they must possess equal number of vertices. Suppose that graph $G_1 = \mathcal{L}(H_{n,p_1})$ and graph $G_2 = \mathcal{L}(H_{n,p_2})$ have the same adjacency spectrum, we show that H_{n,p_1} and H_{n,p_2} are isomorphic.

By Lemma 2.12, $P_{A(G_1)}(\lambda)$ can be computed recursively as follows:

$$\begin{aligned}P_{A(G_1)}(\lambda) = {} & \lambda(P_{A(P_{n-p_1-1})}(\lambda) + P_{A(C_{p_1})}(\lambda)) - (P_{A(C_{p_1})}(\lambda) + P_{A(P_{n-p_1-2})}(\lambda) + 2P_{A(P_{n-p_1-1})}(\lambda) + 2P_{A(P_{p_1-1})}(\lambda)) \\ & - 2(2P_{A(P_{n-p_1-1})}(\lambda) + P_{A(P_{p_1-2})}(\lambda)).\end{aligned}$$

According to Eqs. (2.1) and (2.2), it can be computed by Maple that

$$P_{A(G_1)}(t^{1/2} + t^{-1/2}) = \frac{1}{\sqrt{t}(t-1)}(\psi_1(t) + \psi_2(t) + \psi_3(t) + \psi_4(t) + \psi_5(t)), \tag{3.1}$$

where

$$\begin{aligned}
\psi_1(t) &= 2 - 2t^{\frac{1}{2}} + 2t^{\frac{3}{2}} - 2t^2,\\
\psi_2(t) &= -t^{-\frac{p_1}{2}} + t^{-\frac{p_1}{2}+\frac{1}{2}} + 2t^{-\frac{p_1}{2}+1} + t^{-\frac{p_1}{2}+\frac{3}{2}} + t^{-\frac{p_1}{2}+2},\\
\psi_3(t) &= -t^{\frac{p_1}{2}} - t^{\frac{p_1}{2}+\frac{1}{2}} - 2t^{\frac{p_1}{2}+1} - t^{\frac{p_1}{2}+\frac{3}{2}} + t^{\frac{p_1}{2}+2},\\
\psi_4(t) &= 6t^{-\frac{n}{2}+\frac{p_1}{2}+1} - t^{-\frac{n}{2}+\frac{p_1}{2}+\frac{1}{2}},\\
\psi_5(t) &= t^{\frac{n}{2}-\frac{p_1}{2}+\frac{3}{2}} - 6t^{\frac{n}{2}-\frac{p_1}{2}+1}.
\end{aligned}$$

Similarly, we obtain

$$P_{A(G_2)}(t^{1/2}+t^{-1/2}) = \frac{1}{\sqrt{t}(t-1)}(\psi_1(t)+\varphi_2(t)+\varphi_3(t)+\varphi_4(t)+\varphi_5(t)), \tag{3.2}$$

where $\varphi_2(t)$, $\varphi_3(t)$, $\varphi_4(t)$ and $\varphi_5(t)$ are obtained from $\psi_2(t)$, $\psi_3(t)$, $\psi_4(t)$ and $\psi_5(t)$ by replacing the parameter p_1 with p_2.

Comparing Eqs. (3.1) and (3.2) generates

$$\psi_2(t)+\psi_3(t)+\psi_4(t)+\psi_5(t) = \varphi_2(t)+\varphi_3(t)+\varphi_4(t)+\varphi_5(t). \tag{3.3}$$

Eq. (3.3) implies that

Case 1. $p_1 > \frac{n-1}{2}$, $p_2 > \frac{n-1}{2}$. The largest term of the left side and that of the right side of Eq. (3.3) are $t^{\frac{p_1}{2}+2}$ and $t^{\frac{p_2}{2}+2}$, respectively. Then, $p_1 = p_2$.

Case 2. $p_1 < \frac{n-1}{2}$, $p_2 < \frac{n-1}{2}$. The largest term of the left side and that of the right side of Eq. (3.3) are $t^{\frac{n}{2}-\frac{p_1}{2}+\frac{3}{2}}$ and $t^{\frac{n}{2}-\frac{p_2}{2}+\frac{3}{2}}$, respectively. Then, $p_1 = p_2$.

Case 3. $p_1 = \frac{n-1}{2}$, $p_2 > \frac{n-1}{2}$. The largest term of the left side and that of the right side of Eq. (3.3) are $2t^{\frac{p_1}{2}+2}$ and $t^{\frac{p_2}{2}+2}$, respectively. A contradiction.

Case 4. $p_1 = \frac{n-1}{2}$, $p_2 < \frac{n-1}{2}$. The largest term of the left side and that of the right side of Eq. (3.3) are $2t^{\frac{p_1}{2}+2}$ and $t^{\frac{p_2}{2}+2}$, respectively. A contradiction.

Case 5. $p_1 < \frac{n-1}{2}$, $p_2 = \frac{n-1}{2}$. The largest term of the left side and that of the right side of Eq. (3.3) are $t^{\frac{p_1}{2}+2}$ and $2t^{\frac{p_2}{2}+2}$, respectively. A contradiction.

Case 6. $p_1 > \frac{n-1}{2}$, $p_2 = \frac{n-1}{2}$. The largest term of the left side and that of the right side of Eq. (3.3) are $t^{\frac{p_1}{2}+2}$ and $2t^{\frac{p_2}{2}+2}$, respectively. A contradiction.

Case 7. $p_1 = \frac{n-1}{2}$, $p_2 = \frac{n-1}{2}$. Clearly, $p_1 = p_2$.

Case 8. $p_1 > \frac{n-1}{2}$, $p_2 < \frac{n-1}{2}$. The largest term of the left side and that of the right side of Eq. (3.3) are $t^{\frac{p_1}{2}+2}$ and $t^{\frac{n}{2}-\frac{p_2}{2}+\frac{3}{2}}$, respectively. But, the second largest term of the left side and that of the right side of Eq. (3.3) can not be equal, a contradiction.

Case 9. $p_1 < \frac{n-1}{2}$, $p_2 > \frac{n-1}{2}$. The largest term of the left side and that of the right side of Eq. (3.3) are $t^{\frac{n}{2}-\frac{p_1}{2}+\frac{3}{2}}$ and $t^{\frac{p_2}{2}+2}$, respectively. But, the second largest term of the left side and that of the right side of Eq. (3.3) can not be equal, a contradiction.

Therefore, $p_1 = p_2$, i.e. H_{n,p_1} and H_{n,p_2} are isomorphic. □

Theorem 3.4. *Every lollipop graph $H_{n,p}$ is determined by its Q-spectrum.*

Proof. Suppose that graphs G and $H_{n,p}$ are cospectral with respect to the Q-spectrum, by Lemmas 2.5 and 2.6, $4 < \nu_1(G) = \nu_1(H_{n,p}) \le 6$. Lemma 2.9 implies that G has n vertices, n edges and $\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} d_i'^2$, where d_i, d_i' are degrees of vertex v_i in G and $H_{n,p}$, respectively. Suppose that G has n_i vertices of degree i, for $i = 0, 1, \ldots, \Delta'$, where Δ' is the maximum degree of G. Then

$$\sum_{i=0}^{\Delta'} n_i = n, \tag{3.4}$$

$$\sum_{i=0}^{\Delta'} i n_i = 2n, \tag{3.5}$$

$$\sum_{i=0}^{\Delta'} i^2 n_i = 9 + 4(n-2) + 1. \tag{3.6}$$

Then

$$\sum_{i=0}^{\Delta'} (i^2 - 3i + 2) n_i = 2, \tag{3.7}$$

i.e.,

$$2n_0 + 2n_3 + 6n_4 + \sum_{i=5}^{\Delta'} (i^2 - 3i + 2) n_i = 2. \tag{3.8}$$

Eq. (3.8) implies that

Case 1. $n_0 = 1$, $n_3 = n_4 = \cdots = n_{\Delta'} = 0$. By (3.4) and (3.5), we have $n_1 = -2 < 0$, $n_2 = n + 1$, a contradiction.

Case 2. $n_3 = 1$, $n_0 = n_4 = \cdots = n_{\Delta'} = 0$. By (3.4) and (3.5), we have $n_1 = 1$, $n_2 = n - 2$. Suppose G is not connected, then graph G consists of a lollipop graph and at least one another circuit. Lemma 2.6 implies that $\nu_2(G) = 4$, a contradiction to Lemma 2.8. So, G is connected, and therefore G is a lollipop graph. Suppose $G = H_{n,p_1}$, Lemma 2.10 implies that $\mathcal{L}(H_{n,p_1})$ and $\mathcal{L}(H_{n,p})$ are cospectral with respect to the adjacency matrix. By Theorem 3.3, $p_1 = p$.

Then, G is isomorphic to $H_{n,p}$. $\square$

Acknowledgements

The authors are indebted to the anonymous referees; their useful comments led to this improved version.

References

[1] R. Boulet, B. Jouve, The Lollipop Graph is Determined by its Spectrum, Electronic Journal of Combin. 15 (2008) R74.
[2] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs Theory and Applications, Academic Press, New York, San Francisco, London, 1980.
[3] D. Cvetković, P. Rowlinson, S.K. Simić, Signless Laplacians of finite graphs, Linear Algebra Appl. 423 (2007) 155–171.
[4] M. Doob, W.H. Haemers, The complement of the path is determined by its spectrum, Linear Algebra Appl. 356 (2002) 57–65.
[5] S.M. Fallat, S. Kirkland, S. Pati, Minimizing algebraic connectivity over connected graphs with fixed girth, Discrete Math. 254 (2002) 115–142.
[6] N. Ghareghai, G.R. Omidi, B. Tayfeh-Rezaie, Spectral characterization of graphs with index at most $\sqrt{2+\sqrt{5}}$, Linear Algebra Appl. 420 (2007) 483–489.
[7] C.D. Godsil, B.D. McKay, Constructing cospectral graphs, Aequationes Math. 25 (1982) 257–268.
[8] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, England, 1985.
[9] W.H. Haemers, X.-G. Liu, Y.-P. Zhang, Spectral characterizations of lollipop graphs, Linear Algebra Appl. 428 (2008) 2415–2423.
[10] W.H. Haemers, E. Spence, Enumeration of cospectral graphs, European J. Combin. 25 (2004) 199–211.
[11] X.-G. Liu, Y.-P. Zhang, X.-Q. Gui, The multi-fan graphs are determined by their Laplacian spectra, Discrete Math. 308 (2008) 4267–4271.
[12] R. Merris, Laplacian matrices of graphs: A survey, Linear Algebra Appl. 197&198 (1994) 143–176.
[13] G.R. Omidi, K. Tajbakhsh, Starlike trees are determined by their Laplacian spectrum, Linear Algebra Appl. 422 (2007) 654–658.
[14] Q.-J. Ren, J. Tian, D.-X. Zhang, F.-Y. Wang, The Quasi-Laplacian Spectrum of Graphs, Math. Theory Appl. 19 (3) (1999) 104–107.
[15] X.-L. Shen, Y.-P. Hou, Y.-P. Zhang, Graph Z_n and some graphs related to Z_n are determined by their spectrum, Linear Algebra Appl. 404 (2005) 58–68.
[16] S.K. Simić, Z. Stanić, Q-integral graphs with edge-degrees at most five, Discrete Math. 308 (2008) 4625–4634.
[17] E.R. van Dam, W.H. Haemers, Which graphs are determined by their spectrum? Linear Algebra Appl. 373 (2003) 241–272.
[18] W. Wang, C.-X. Xu, On the spectral characterization of T-shape trees, Linear Algebra Appl. 414 (2006) 492–501.
[19] W. Wang, C.-X. Xu, Note: The T-shape tree is determined by its Laplacian spectrum, Linear Algebra Appl. 419 (2006) 78–81.