Asymptotic enumeration theorems for the numbers of spanning trees and Eulerian trails in circulant digraphs and graphs*

ZHANG Fuji (张福基)

(Department of Mathematics, Xiamen University, Xiamen 361005, China)

and YONG Xuerong (永学荣)

(Department of Computer Science, Hong Kong University of Science & Technology, Hong Kong, China;

Department of Mathematics, Xinjiang University, Urumqi 361005, China)

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Abstract The asymptotic properties of the numbers of spanning trees and Eulerian trails in circulant digraphs and graphs are studied. Let $C(p, s_1, s_2, \cdots, s_k)$ be a directed circulant graph. Let $T(C(p, s_1, s_2, \cdots, s_k))$ and $E(C(p, s_1, s_2, \cdots, s_k))$ be the numbers of spanning trees and of Eulerian trails, respectively. Then

$$\lim \frac{1}{k} \sqrt[p]{T(C(p, s_1, s_2, \cdots, s_k))} = 1, \qquad \lim \frac{1}{k!} \sqrt[p]{E(C(p, s_1, s_2, \cdots, s_k))} = 1, \qquad p \to \infty.$$

Furthermore, their line digraph and iterations are dealt with and similar results are obtained for undirected circulant graphs.

Keywords: **asymptotic enumeration, Eulerian trails, graph, circulant.**

With the development of random graph theory many mathematicians have paid more attention to a class of graphs than to a single graph in the study of the global properties or the properties that almost all graphs possess. In this paper, we will focus on such kinds of properties of circulant directed and undirected graphs.

Directed graphs, to some extent, give a general model of communication networks. It was first studied in ref. [1]. A recent survey can be found in ref. [2]. Their line digraph iterations (e.g. de Brujin graph, Kautz graph, etc.) have been widely used in the area of designing networks. The reader may refer to refs. [3, 4] for recent development. On this topic, the numbers of spanning trees and Eulerian trails are two basic parameters to be considered. Profound studies have been made on the different networks with different enumerations[5—7]. These facts motivated us to study the numbers of spanning trees and the Eulerian trails. We have obtained the asymptotic theorems for the circulant digraphs with n vertices and k out-degrees. We note that for a fixed k and p the circulant digraph with p vertices and out-degree k may have different numbers of spanning trees and Eulerian trails. In other words in this case the role of jump sequences cannot be neglected. Our theorems show that when the vertices are sufficiently large in

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number, the asymptotic behavior of the numbers of spanning trees and Eulerian trails of the circulant digraphs depends only on their out-degree and scarecely depend on the structure of G. Furthermore, a similar result is valid forundirected cases. This phenomenon is interesting, but not unique. For the asymptotic enumeration on perfect matching of k-regular bipartite graph, Schrijver and Valiant posed in 1980 a conjecture[8] which is formally similar to ours. But so far only a special case of $k=3$ has been solved. This is the first nontrivial case.

1 Preliminaries

First, we recall some definitions. The circulant digraph $C(p, s_1, s_2, \cdots, s_k)$ is defined as the digraph with p vertices labeled with integer modulo p such that for each vertex there are k arcs from i to $i+s_1$, $i+s_2$, $\cdots$, $i+s_k \pmod p$ where $1 \leqslant s_1 < s_2 < \cdots < s_k \leqslant p-1$.

The line digraph $L(D)$ of digraph $D=(V, A)$ is defined as follows: the vertices of $L(D)$ represent the arcs of D; that is, $V(L(D)) = \{uv \mid (u, v) \in A\}$ and the vertices uv is adjacent to the vertex wz if $v=w$. If D is a k-regular digraph (i.e. $D^+(v) = D^-(v) = k$ for all vertices $v \in V$, where $D^+(v) = |\{(u:(v,u) \in A)\}|$ and $D^-(v) = |\{u:(u,v) \in A\}|$ denote the out-degree and in-degree of vertex v, respectively). Then its line digraph is also k-regular with $|V(L(D))| = k|V(D)|$ and $|A(L(D))| = k^2|V(D)|$. Furthermore we can define the sequence of iterated line digraph as follows:

$$L^0(D) = D, \ L^{i+1}(D) = L(L^i(D)), \ i \geqslant 0.$$

One can see that $L^i(D)$ is also a k-regular digraph.

In order to prove our main results, we need some lemmas. We would like to point out that Lemmas 1 and 2 are known results.

Lemma 1[9]. *The characteristic polynomial of the circulant digraph is*

$$\chi(C(p, s_1, s_2, \cdots, s_k), \lambda) = \prod_{j=0}^{p-1} (\lambda - \varepsilon^{s_1 j} - \varepsilon^{s_2 j} - \cdots - \varepsilon^{s_k j}),$$

where $\varepsilon = \exp \dfrac{2\pi\sqrt{-1}}{p}$.

Lemma 2[10]. *Let D be a regular directed graph and let $\chi(D, \lambda)$ be its characteristic polynomial. Then the number of spanning trees of D is*

$$T(D) = \chi'(D, k) = \frac{\mathrm{d}}{\mathrm{d}\lambda} X(D, \lambda) \mid_{\lambda = k}.$$

Since $C(p, s_1, s_2, \cdots, s_k)$ is k-regular (each vertex has the same in-(out-) degree k). By Lemmas 1 and 2, taking the derivative of $\chi(C(p, s_1, s_2, \cdots, s_k), \lambda)$ with respect to λ and evaluating at $\lambda = k$ we have

Lemma 3. *The number spanning trees of a directed graph is*

$$T(C(p, s_1, s_2, \cdots, s_k)) = \prod_{j=1}^{p-1} (k - \varepsilon^{s_1 j} - \varepsilon^{s_2 j} - \cdots - \varepsilon^{s_k j}),$$

where $\varepsilon = \exp \dfrac{2\pi\sqrt{-1}}{p}$.

Lemma 4. *Let $C(p, s_1, s_2, \cdots, s_k)$ be a circulant digraph. Let polynomial* $f(z) = \sum_{i=0}^{s_1-1} z^i + \sum_{i=0}^{s_2-1} z^i + \cdots + \sum_{i=0}^{s_k-1} z^i$ *have the roots $\alpha_1, \alpha_2, \cdots, \alpha_{s_k-1}$. Then*

$$T(C(p, s_1, s_2, \cdots, s_k)) = p\,\frac{(-1)^{(p-1)(s_k-1)}}{f(1)}\prod_{j=1}^{s_k-1}(1-\alpha_j^p), \tag{1}$$

where $f(1) = s_1 + s_2 + \cdots + s_k - k$.

Proof. By Lemma 3 we have

$$\begin{aligned} T(C(p, s_1, s_2, \cdots, s_k)) &= \prod_{j=1}^{p-1}(1-\varepsilon^{s_1 j} + 1 - \varepsilon^{s_2 j} + \cdots + 1 - \varepsilon^{s_k j}) \\ &= \prod_{j=1}^{p-1}(1-\varepsilon^j)\prod_{j=1}^{p-1}\Big(\sum_{i=0}^{s_1-1}\varepsilon^{ij} + \sum_{i=0}^{s_2-1}\varepsilon^{ij} + \cdots + \sum_{i=0}^{s_k-1}\varepsilon^{ij}\Big) \\ &= p\prod_{j=1}^{p-1}\Big(\sum_{i=0}^{s_1-1}\varepsilon^{ij} + \sum_{i=0}^{s_2-1}\varepsilon^{ij} + \cdots + \sum_{i=0}^{s_k-1}\varepsilon^{ij}\Big). \end{aligned}$$

Since

$$\prod_{j=1}^{p-1}(x-\varepsilon^j) = \sum_{i=0}^{p-1}x^i, \tag{2}$$

evaluating at $x=1$ we have

$$\prod_{j=1}^{p-1}(1-\varepsilon^j) = p.$$

Furthermore by the definition of $f(z)$ and using (2) again we have

$$\begin{aligned} T(C(p, s_1, s_2, \cdots, s_k)) &= p\prod_{j=1}^{p-1}f(\varepsilon^j) = p\prod_{j=1}^{p-1}(\varepsilon^j-\alpha_1)\prod_{j=1}^{p-1}(\varepsilon^j-\alpha_2)\cdots\prod_{j=1}^{p-1}(\varepsilon^j-\alpha_{s_k-l}) \\ &= (-1)^{(p-1)(s_k-1)}p\Big(\sum_{j=0}^{p-1}\alpha_1^j\Big)\Big(\sum_{j=0}^{p-1}\alpha_2^j\Big)\cdots\Big(\sum_{j=0}^{p-1}\alpha_{s_k-1}^j\Big) \\ &= (-1)^{(p-1)(s_k-1)}p\prod_{j=1}^{s_k-1}\frac{1-\alpha_j^p}{1-\alpha_j} = (-1)^{(p-1)(s_k-1)}\frac{p}{f(1)}\prod_{j=1}^{s_k-1}(1-\alpha_j^p). \end{aligned}$$

Lemma 4 is thus proved.

Lemma 5. *Let*

$$f(z) = \sum_{i=0}^{s_1-1}z^i + \sum_{i=0}^{s_2-1}z^i + \cdots + \sum_{i=0}^{s_k-1}z^i,$$

where $1 \leqslant s_1 < s_2 < \cdots < s_k$.

If the greatest common divisor $(s_1, s_2, \cdots, s_k) = 1$, then the roots of $f(z)$ satisfy

$$|\alpha_i| > 1,\ i = 1, 2, \cdots, s_k - 1.$$

Proof. Noting $f(1) \neq 0$ and

$$(z-1)f(z) = z^{s_1} + z^{s_2} + \cdots + z^{s_k} - k,$$

for each root α_j of $f(z)$ $\alpha_j \neq 1$, we have

$$\alpha_j^{s+1} + \alpha_j^{s+2} + \cdots + \alpha_j^{s+k} = k. \tag{3}$$

Recalling that, if $|\alpha_j| < 1$, we have

$$|\alpha_j^{s+1} + \alpha_j^{s+2} + \cdots + \alpha_j^{s+k}| \leqslant |\alpha_j|^{s+1} + |\alpha_j|^{s+2} + \cdots + |\alpha_j^{s+k}| < k.$$

This is a contradiction. Therefore $|\alpha_j| \geqslant 1$.

Since $\alpha_i \neq 1$, if $|\alpha_j| = 1$, we have

$$\alpha_j = \cos\varphi_j + i\sin\varphi_j,\ \sin\varphi_j \neq 0,\ j = 1, 2, \cdots, s_k - 1.$$

By (3) the above deduces to

$$\cos s_1\varphi_j + \cos s_2\varphi_j + \cdots + \cos s_k\varphi_j = k.$$

Hence

$$\cos s_l\varphi_j = 1,\ l = 1, 2, \cdots, k,$$

which means α_j is a unit root of degree $m\,(m > 1)$ and m is a common divisor of s_i. This contradicts the assumption $(s_1, s_2, \cdots, s_k) = 1$. Thus $|\alpha_j| > 1,\ j = 1, 2, \cdots, s_k - 1$. The lemma is proved.

2 Main results

Now we prove our main results.

Theorem 1. *For the circulant digraph $C(p, s_1, s_2, \cdots, s_k)$, if $(s_1, s_2, \cdots, s_k) = 1$, then*

$$T(C(p, s_1, s_2, \cdots, s_k)) \sim \frac{pk^p}{f(1)},\ p \to \infty. \tag{4}$$

Proof. Let

$$\sigma_1(p) = \sum_{j=1}^{s_k-1} \alpha_j^p,\ \sigma_2(p) = \sum_{1 \leqslant i < j \leqslant s_k-1} \alpha_i^p\alpha_j^p,$$

$$\sigma_3(p) = \sum_{1 \leqslant i < j < r \leqslant s_k-1} \alpha_i^p\alpha_j^p\alpha_r^p, \cdots,$$

$$\sigma_{s_k-1}(p) = \prod_{j=1}^{s_k-1} \alpha_j^p$$

be the elementary symmetric polynomials of $\alpha_1^p, \alpha_2^p, \cdots, \alpha_{s_k-1}^p$. Then we have[11]

$$\prod_{j=1}^{s_{n-1}} (z - \alpha_j^p) = z^{s_k-1} - \sigma_1(p)z^{s_k-2} + \sigma_2(p)z^{s_k-3} - \sigma_3(p)z^{s_k-4} + \cdots + (-p)^{s_k-1}\sigma_{s_k-1}(p).$$

Taking $z = 1$, we have

$$\prod_{j=1}^{s_k-1} (1 - \alpha_j^p) = 1 - \sigma_1(p) + \sigma_2(p) - \sigma_3(p) + \cdots + (-1)^{s_k-1}\sigma_{s_k-1}(p).$$

Noting that $\sigma_i(1),\ i = 1, 2, \cdots, s_k - 1$ are the elementary symmetry polynomials of $\alpha_1, \alpha_2, \cdots, \alpha_{s_k-1}$, by definition of $f(z)$ and putting $z = 0$, we have

$$\sigma_{s_k-1}(1) = (-1)^{s_k-1}\alpha_1\alpha_2\cdots\alpha_{s_k-1} = k.$$

Hence

$$\sigma_{s_k-1}(p) = (-1)^{(s_k-1)p}k^p.$$

Since by Lemma 5 for $i < s_k - 1$

$$\frac{\sigma_i(p)}{\sigma_{s_k-1}(p)} \to 0,\ p \to \infty,$$

by Lemma 4 we have

$$\frac{T(C(p, s_1, s_2, \cdots, s_k))}{\frac{1}{f(1)}pk^p}$$

$$= \frac{(-1)^{(p-1)(s_k-1)+(s_k-1)p}(1 - \sigma_1(p) + \sigma_2(p) + \cdots + (-1)^{s_k-1}\sigma_{s_k-1}(p))}{\sigma_{s_k-1}(p)}$$

$$=\frac{(-1)^{s_k-1}}{\sigma_{s_k-1}(p)}-\frac{(-1)^{s_k-1}\sigma_1(p)}{\sigma_{s_k-1}(p)}+\frac{(-1)^{s_k-1}\sigma_2(p)}{\sigma_{s_k-1}(p)}$$

$$-\frac{(-1)^{s_k-1}\sigma_3(p)}{\sigma_{s_k-1}(p)}+\cdots+-\frac{(-1)^{s_k-1}(-1)^{s_k-1}\sigma_{s_k-1}(p)}{\sigma_{s_k-1}(p)}\rightarrow 1,\ p\rightarrow\infty.$$

The proof is thus completed.

The following result is similar to the conjecture of Schrijver and Valiant for the number of perfect matchings of k-regular biparite graph which can be obtained by Theorem 1.

Theorem 2. *For the circulant digraph* $C(p, s_1, s_2, \cdots, s_k)$

$$\lim \frac{1}{k}\{T(C(p, s_1, s_2, \cdots, s_k))\}^{\frac{1}{p}} = 1,\ p\rightarrow\infty.$$

Proof. By Theorem 1 and $f(1) = s_1 + s_2 + \cdots + s_k - k \geqslant 1$, when k is fixed we have $f(1)^{\frac{1}{p}}\rightarrow 1,\ p\rightarrow\infty$.

For the number of Eulerian trails we have

Lemma 6(ref.[5] or Lemma 5 of ref.[6]). *Let D be a k-regular digraph with p vertices. Then the number of Eulerian trails is given by*

$$E(D) = T'(D)((k-1)!)^p,$$

where $T'(D)$ *is the number of directed spanning trees with any vertex as its root.*

Theorem 3. *Let* $C(p, s_1, s_2, \cdots, s_k)$ *be the circulant diagraph. Then its number of Eulerian trails has the following asymptotic property:*

$$\lim_{p\rightarrow\infty}\frac{1}{k!}\{E(C(p, s_1, s_2, \cdots, s_k))\}^{\frac{1}{p}} = 1.$$

Proof. For the k-regular directed graph $C(p, s_1, s_2, \cdots, s_k)$, we have

$$T'(C(p, s_1, s_2, \cdots, s_k)) = \frac{1}{p}T(C(p, s_1, s_2, \cdots, s_k)).$$

where $T(D)$ is the number of spanning of D. By Theorem 1 and Lemma 6

$$T'(C(p, s_1, s_2, \cdots, s_k)) \sim \frac{1}{f(1)}(k)^p,\ p\rightarrow\infty,$$

$$E(C(p, s_1, s_2, \cdots, s_k)) \sim \frac{1}{f(1)}(k!)^p,\ p\rightarrow\infty.$$

The rest of the proof is similar to the proof of Theorem 2.

Now we turn to the iterated line digraph.

Lemma 7(Theorem 1 of ref.[6]). *Let D be the k-regular digraph with p vertices. Let* $L^r(D)$ *be its rth iterated line digraph. Then the number of spanning trees of* $L^r(D)$*, say* $T(L^r(D))$*, is*

$$T(L^r(D)) = k^{pkr-p}T(D).$$

Using the approaches in proving Theorem 2.3, it is not difficult to get the following

Theorem 4. *Let* $L^r(C(p, s_1, s_2, \cdots, s_k))$ *be the r-th iterated line digraph of* $C(p, s_1, s_2, \cdots, s_k)$. *Then the numbers of spanning trees and Eulerian trails have the following asymptotic properties:*

$$\lim_{p\rightarrow+\infty}\frac{1}{k^{kr}}\{T(L^r(C(p, s_1, s_2, \cdots, s_k)))\}^{\frac{1}{p}} = 1,$$

$$\lim_{p\rightarrow+\infty}\frac{1}{(k!)^{kr}}\{E(L^r(C(p, s_1, s_2, \cdots, s_k)))\}^{\frac{1}{p}} = 1,$$

where $T(D)$ is the number of spanning tress of D.

3 Undirected circulant graphs

Like directed graph, the undirected ones have also been widely used in computer networks[2,3]. Now we discuss the asymptotic properties. Because proofs are similar, we omit the details. For undirected case, the graph $C(p, s_1, s_2, \cdots, s_k)$ with $1 \leqslant s_1 < s_2 < \cdots < s_k \leqslant \frac{p}{2}$ is $2k$-regular when $s_k < \frac{p}{2}$ and $(2k-1)$-regular when $s_k = \frac{p}{2}$. This means that when $s_k < \frac{p}{2}$, $C(p, s_1, s_2, \cdots, s_k)$ is k-regular.

Lemma 8[9]. *The characteristic polynomial of the circulant graph is*

$$\chi(C(p, s_1, s_2, \cdots, s_k), \lambda) = \begin{cases} \prod_{j=0}^{p-1} (\lambda - \varepsilon^{s_1 j} - \cdots - \varepsilon^{s_k j} - \varepsilon^{(p-s_1)j} - \cdots - \varepsilon^{(p-s_k)j}), & s_k < \frac{p}{2}; \\ \prod_{j=0}^{p-1} (\lambda - \varepsilon^{s_1 j} - \cdots - \varepsilon^{s_k j} - \varepsilon^{(p-s_1)j} - \cdots - \varepsilon^{(p-s_{k-1})j}), & s_k = \frac{p}{2}. \end{cases}$$

Lemma 9[9]. *For a k-regular graph Γ with p vertices, the number of spanning trees $T(\Gamma)$ is given by*

$$T(\Gamma) = p^{-1}\chi'(\Gamma, k),$$

where $\chi'(\Gamma, k)$ is the value of derivative function of $\chi(\Gamma, \lambda)$ at $\lambda = k$.

Lemma 10. *The number of spanning trees of $C(p, s_1, s_2, \cdots, s_k)$ is*

$$T(C(p, s_1, s_2, \cdots, s_k)) = \begin{cases} \frac{1}{p}\prod_{j=1}^{p-1} (2k - \varepsilon^{s_1 j} - \cdots - \varepsilon^{s_k j} - \varepsilon^{(p-s_1)j} - \cdots - \varepsilon^{(p-s_k)j}), & s_k < \frac{p}{2}; \\ \frac{1}{p}\prod_{j=1}^{p-1} (2k - 1 - \varepsilon^{s_1 j} - \cdots - \varepsilon^{s_k j} - \varepsilon^{(p-s_1)j} - \cdots - \varepsilon^{(p-s_{k-1})j}), & s_k = \frac{p}{2}. \end{cases}$$

Lemma 11. *Let*

$$f(x) = \begin{cases} \sum_{i=0}^{s_1-1} z^i + \sum_{i=0}^{s_2-1} z^i + \cdots + \sum_{i=0}^{s_k-1} z^i + \sum_{i=0}^{p-s_1-1} z^i + \sum_{i=0}^{p-s_2-1} z^i + \cdots + \sum_{i=0}^{p-s_k-1} z^i, & s_k < \frac{p}{2}; \\ \sum_{i=0}^{s_1-1} z^i + \sum_{i=0}^{s_2-1} z^i + \cdots + \sum_{i=0}^{s_k-1} z^i + \sum_{i=0}^{p-s_1-1} z^i + \cdots + \sum_{i=0}^{p-s_{k-1}-1} z^i, & s_k = \frac{p}{2}, \end{cases}$$

and let $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots, \alpha_{p-s_1-1}$ be the roots of $f(z)$. Then

$$T(C(p, s_1, s_2, \cdots, s_k)) = \frac{(-1)^{(p-1)(p-s_1-1)}}{f(1)} \prod_{i=0}^{p-s_1-1} (1 - \alpha_j^p).$$

Lemma 12. *Let $f(z)$ be defined in Lemma 11. If the greatest common divisor $(s_1, s_2, \cdots, s_k) = 1$, then all the roots α_j, $j = 1, 2, \cdots, p - s_1 - 1$, of $f(z)$ satisfy $|\alpha_j| > 1$.*

Theorem 5. *If $(s_1, s_2, \cdots, s_k) = 1$, then*

$$T(C(s_1, s_2, \cdots, s_k)) \sim \begin{cases} \frac{(2k)^p}{f(1)}, & \text{for } s_k < p/2, \\ \frac{(2k-1)^p}{f(1)}, & \text{for } s_k = p/2. \end{cases}$$

Theorem 6. *If $(s_1, s_2, \cdots, s_k) = 1$, then*

$$\lim_{p\to\infty}\frac{1}{2k}\{T(C(p,s_1,s_2,\cdots,s_k))\}^{\frac{1}{p}}=1,\ \text{for } s_k<\frac{p}{2},$$

$$\lim_{p\to\infty}\frac{1}{2k-1}\{T(C(p,s_1,s_2,\cdots,s_k))\}^{\frac{1}{p}}=1,\ \text{for } s_k=\frac{p}{2}.$$

The proofs follow from the fact that $1\leqslant f(1)\leqslant 2kp$. Now we consider the asymptotic properties of the line iterations of $C(s_1,s_2,\cdots,s_k)$.

In the following we assume that Γ is d-regular and the line graph is denoted by $L(\Gamma)$.

Lemma 13[9]. *Let $T(\Gamma)$ be the numbers of spanning trees of Γ. Then the number of spanning trees of its line graph is*

$$T(L(\Gamma))=2^{q-p+1}d^{q-p-1}T(\Gamma),$$

where p and q are the numbers of vertices and edges of Γ.

Theorem 7. *Let Γ be a d-regular graph with p vertices. Let $T(\Gamma)$ be the number of its spanning trees. Then the number of spanning irees of the r-iteration of its line graph is*

$$T(L^r(\Gamma))=(2d)^{\frac{1}{2}pd\left(\prod\limits_{i=1}^{n}(2^{i-1}d-2^i+1)-1\right)}(2/d)^rT(\Gamma).$$

Proof. It is easy to see that $L(\Gamma)$ is $(2d-2)$-regular graph with $\frac{pd}{2}$ vertices and $\frac{pd(d-1)}{2}$ edges. Making use of induction, we verify readily that $L^n(T)$ is $(2^nd-2^{n+1}-2)$-regular. Assume that the number of vertices of $L^k(\Gamma)$ is v_k. Then we have

$$v_{n+1}=\frac{1}{2}pd\prod_{i=1}^{n}(2^{i-1}d-2^n+1).$$

Therefore, using Lemma 13 repeatedly, we have

$$T(L^r(\Gamma))=2^{v_{r+1}-v_r+1}d^{v_{r+1}-v_r-1}2^{v_r-v_{r-1}+1}d^{v_r-v_{r-1}-1}\cdots 2^{v_2-v_1+1}d^{v_2-v_1-1}T(\Gamma)\cdot$$

$$(2d)^{v_{r+1}-v_1}\left(\frac{2}{d}\right)^rT(\Gamma).$$

Replacing n in v_{n+1} by r, using relation $v_1=\frac{1}{2}pd$, the proof of Theorem 7 is completed.

Theorem 8. *Let $\Gamma=C(p,s_1,s_2,\cdots,s_k)$ be a circulant graph with greatest common divisor $(s_1,s_2,\cdots,s_k)=1$. Then*

$$\sqrt[p]{T(L^r(C(p,s_1,s_2,\cdots,s_k)))}\simeq d(2d)^{\frac{1}{2}d\left(\prod\limits_{i=1}^{n}(2^{i-1}d-2^i+1)-1\right)},\ p\to\infty,$$

where

$$d=\begin{cases}2k, & \text{if } s_k\neq\frac{p}{2},\\ 2k-1, & \text{if } s_k=\frac{p}{2}.\end{cases}$$

Proof. Since r is fixed, we have

$$\left\{\left(\frac{d}{2}\right)^r\right\}^{\frac{1}{p}}\to 1,\ p\to\infty.$$

The proof follows directly from Theorems 5 and 7.

Remark 1. If $(s_1,s_2,\cdots,s_k)\neq 1$, Theorem 1 (hence Theorems 3, 4, 8) does not hold. In fact, we may choose p_n such that $p_n\to\infty$, $n\to\infty$, and the graph $C(p_n,s_1,s_2,\cdots,s_n)$ is not connected. This gives

$$T(C(p_n,s_1,s_2,\cdots,s_n))=0$$

for all integer n.

Remark 2. In ref. [12], we discussed the case $C(p, 1, q)$ by using different approaches. But those methods cannot be applied to the general case.

We would like to pose the following two open problems at the end of the paper.

Problem 1. *How to consider the asymptotic properties of the numbers of spanning trees and Eulerian trails for d-circulant digraph where d-circulant digraph is a graph whose adjacency matrix is a d-circulant matrix*[13]? *A special case of this graph is called generalized de Brujin graph, which was introduced by Imas and Itoh*[14,15]. *The number of spanning trees and Eulerian trails was considered by Li Xue-Liang and one of the authors of this paper (see* [16]).

Problem 2. *How to determine the number of Eulerian trails of the iterated line graph of a circulant graph*?

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