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Short Communication: Two Properties of Diagonally Dominant Matrices

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A property of strictly diagonally dominant matrices and a generalization of a Varga's bound for $\|A^{-1}\|_\infty$ to the case $\|A^{-1}B\|_\infty$ are given and the two-sided bounds for the determinants of strictly diagonally dominant matrices are derived.

KEY WORDS diagonally dominant matrix; determinant; norm

1. Introduction and notation

Let $A = (a_{ij})$ be an arbitrary $n \times n$ complex matrix. If

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, i = 1, 2, \ldots, n$$

then A is called a row strictly diagonally dominant matrix (the column is defined similarly). Such matrices occur frequently in mathematics and engineering. It is well known that they are important when studying M-matrices [1]. Recently, some numerical properties of these matrices have been obtained [2, 3] and applications of them are emerging [4, 5].

Throughout this short communication we denote the jth entry of the vector x by $(x)_j$; superscripts T and H denote the transpose and conjugate transpose, respectively. For a vector $x = (x_1, x_2, \ldots, x_n)$ and a matrix A, $\|x\|_\infty$, and $\|A\|_\infty$ means $\|x\|_\infty = \max |x_i|$, and $\|A\|_\infty = \sup_{\|x\|_\infty=1} \|Ax\|_\infty$, respectively.

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2. Main results

Lemma 2.1. *Let $a^T = (a_1, a_2, \ldots, a_n) \neq 0$ be a real vector. If, for an arbitrary vector $x^T = (x_1, x_2, \ldots, x_n) \neq 0$,*

$$a^T x = 0$$

implies that $|x_i| < \|x\|_\infty$ (i is fixed), then

$$|a_i| > \sum_{j \neq i} |a_j|$$

and vice versa.

Proof Without loss of generality we suppose that $a_j > 0 (j = 1, 2, \ldots, n)$. We need only show the result for $i = 1$ and use induction on n (the dimension of vector a). For $n = 2$, it yields

$$a_1 x_1 = -a_2 x_2$$

This gives rise to $a_1 > a_2$, provided that

$$|x_1| < |x_2| = \|x\|_\infty$$

Now, assuming that this is true for $n = k - 1$, we are able to verify this for $n = k$. Since the vector $x \neq 0$ is arbitrary, it is easy to see that

$$a_1 > a_j, \qquad j = 2, 3, \ldots, k$$

Let $\|x\|_\infty = |x_k| > |x_1|$, then since

$$a_1 x_1 + a_2 x_2 + \ldots + a_k x_k = 0$$

we have

$$(a_1 - a_2) x_1 + a_2 y + a_3 x_3 + \ldots + a_k x_k = 0$$

with $y = x_1 + x_2$. Hence,

$$(a_1 - a_2) x_1 + a_3 t + a_4 x_4 + \ldots, a_k x_k = 0$$

with

$$t = \frac{a_2(x_1 + x_2)}{a_3} + x_3$$

this leads to $b^T z = 0$, where $b = (a_1 - a_2, a_3, \ldots, a_k)^T$, and $z = (x_1, t, x_4, \ldots, x_k)^T$. This is the case where $n = k - 1$. Therefore, by $|x_1| < \|x\|_\infty = \|z\|_\infty$ and the assumption of induction, we have

$$a_1 - a_2 > a_3 + \ldots + a_k$$

That is

$$a_i > \sum_{j=2} a_j$$

which, of necessity, completes the proof.

Conversely, since $a^T x = 0$, we have

$$a_1 x_1 = -\sum_{j=2}^{n} a_j x_j$$

Hence, taking into account that $a_j > 0$, $j = 1, 2, \ldots, n$, this yields

$$\begin{aligned} a_1|x_1| &\leq \sum_{j=2}^{n} a_j |x_j| \\ &\leq \left(\sum_{j=2}^{n} a_j \right) \|x\|_\infty \end{aligned}$$

Noticing that $a_1 > \sum_{j=2}^{n} a_j$, the previous inequality implies that

$$|x_j| < \|x\|_\infty$$ ■

Theorem 2.1. *An $n \times n$ matrix A is row strictly diagonally dominant if and only if for any $x^T = (x_1, x_2, \ldots, x_n) \neq 0$ the equality $(Ax)_j = 0$ implies*

$$|x_j| < \|x\|_\infty$$

Proof Apply Lemma 2.1. ■

Lemma 2.2. *Let $B = (b_{ij})$ be an $n \times m$ matrix and $A = (a_{ij})$ be an $n \times n$ row strictly diagonally dominant matrix. Then*

$$\|A^{-1}B\|_\infty \leq \max_i \frac{\sum_{j=1}^{m} |b_{ij}|}{|a_{ii}| - \sum_{j \neq i} |a_{ij}|}$$

Proof Since there exists an m-dimensional vector $x = (x_1, x_2, \ldots, x_m)^T$ ($\|x\|_\infty = 1$) such that

$$\|A^{-1}B\|_\infty = \|A^{-1}Bx\|_\infty = \|y\|_\infty = |y_{i_0}|$$

where $A^{-1}Bx = y$ and $y = (y_1, y_2, \ldots, y_n)^T$, we have

$$\sum_{j=1}^{m} b_{i_0 j} x_j = a_{i_0 i_0} y_{i_0} + \sum_{j \neq i_0} a_{i_0 j} y_j$$

Hence,

$$\sum_{j=1}^{m} |b_{i_0 j}| \max_{1 \leq j \leq m} |x_j| \geq \left(|a_{i_0 i_0}| - \sum_{j \neq i_0} |a_{i_0 j}| \right) |y_{i_0}|$$

implying

$$|y_{i_0}| \leq \frac{\sum_{j=1}^{m} |b_{i_0 j}|}{|a_{i_0 i_0}| - \sum |a_{i_0 j}|} \max_{1 \leq j \leq n} |x_j|$$

Taking into account that $|y_{i_0}| = \|A^{-1}B\|_\infty$, and $\|x\|_\infty = \max_{1 \leq j \leq n} |x_j| = 1$, this completes the proof. ■

In the case $B = I$, the result of Lemma 2.2 reduces to

$$\|A^{-1}\|_\infty \le \max_{1\le i\le n} \frac{1}{|a_{ii}| - \sum_{j\ne i} |a_{ij}|}$$

which is the R.S. Varga's bound [6].

Theorem 2.2. *Let $A = (a_{ij})$ be an $n \times n$ row strictly diagonally dominant matrix. Then for the determinant of A the following two-sided bounds hold;*

$$\prod_{i=1}^{n} \left(|a_{ii}| - r_i \sum_{j\ne i+1}^{n} |a_{ij}| \right) \le |\det A| \le \prod_{i=1}^{n} \left(|a_{ii}| + r_i \sum_{j\ne i+1}^{n} |a_{ij}| \right)$$

where

$$\begin{cases} r_i = \max\limits_{i+1\le k\le n} \dfrac{|a_{ki}|}{|a_{kk}| - \sum_{j=i+1}^{n} |a_{kj}|} \le 1, \qquad (i = 1, 2, \ldots, n-1) \\ r_n = 0 \end{cases}$$

Proof Let

$$A = \begin{pmatrix} a_{11} & b^H \\ c & A_1 \end{pmatrix}$$

where $b^H = (a_{12}, a_{13}, \ldots, a_{1n})$, $c = (a_{21}, a_{31}, \ldots, a_{n1})^H$. Then

$$\det A = (a_{11} - b^H A_1^{-1} c) \det A_1$$

Since A_1 is also a row strictly diagonally dominant matrix, by Lemma 2.2 we have

$$\begin{aligned} |b^H A_1^{-1} c| &\le \|b\|_1 \, \|A_1^{-1} c\|_\infty \\ &\le \sum_{j=2}^{n} |a_{kj}| \max_{2\le k\le n} \frac{a_{k1}}{|a_{kk}| - \sum_{j=2,\, j\ne k}^{n} |a_{kj}|} \\ &= \sum_{j=2}^{n} |a_{kj}| \cdot r_1 \end{aligned}$$

Hence, by

$$(|a_{11}| - |b^H A_1^{-1} c|)\, |\det A_1| \le |\det A| \le (|a_{11}| + |b^H A_1^{-1} c|)\, |\det A_1|$$

$\rho(A_i^{-1}) = 1$?

we have

$$\left(|a_{11}| - r_1\sum_{j=2}^{n}|a_{ij}|\right)|\det A_1| \le |\det A| \le \left(|a_{11}| + r_1\sum_{j=2}^{n}|a_{1j}|\right)|\det A_1|$$

In view of the above inequality, applying these bounds now to A_1 and so on proves Theorem 2.2. ■

Theorem 2.1 gives a theoretical result on diagonally dominant matrices and Theorem 2.2 can be used to estimate the lower bounds for the number of spanning trees of some graphs (see [7]) and the lower bounds for the smallest singular values of such matrices (see [8]).

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REFERENCES

1. A. Berman *et al. Nonnegative Matrices in the Mathematical Sciences,* Academic Press, New York, 1979.
2. C. R. Johnson. A Gerschgorin-type lower bound for the smallest singular value, *Lin. Alg. Appl.,* 112, 1–7, 1989.
3. You Zhao-Yong *et al.* The distributions of eigenvalues of conjugate generalized diagonally dominant matrices, *J. Math. Research and Exposition* (in Chinese), 9(2), 309–310, 1989.
4. Yong Xue-Rong, Applying H-matrices to determine the positive stability of some matrices, *J. Engine. Math.,* 8(2), 163–169, 1991.
5. Shi Zhong-Ci. A note on nonsingularity of the interpolating matrix of cubic spline, *Math. Numer. Sinica,* 5(2), 195–204, 1983.
6. R. S. Varga. On diagonal dominance arguments for bounding $\|A^{-1}\|_\infty$, *Lin. Alg.Appl.,* 14, 211–217, 1976.
7. N. Biggs. *Algebraic Graph Theory,* Cambridge, 1976.
8. Y. P. Hong *et al.* A lower bound for the smallest singular value, *Lin. Alg. Appl.,* 172, 27–32, 1992.