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Disproof of a conjecture on the existence of the path-recursive period for a connected graph ☆

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Abstract

A sufficient and necessary condition on the existence of the path-recursive period for a graph has been established in this paper. This disproves the conjecture proposed in [R. Shi, Path polynomials of a graph, Linear Algebra Appl. 236 (1996) 181–187]. Some results presented in [R. Shi, Path polynomials of a graph, Linear Algebra Appl. 236 (1996) 181–187; R.B. Bapat, A.K. Lal, Path-positive graphs, Linear Algebra Appl. 149 (1991) 125–149] have also been included. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

All graphs considered here are simple and undirected.

Let G denote a graph with vertex set $\{v_1, \ldots, v_n\}$. Its adjacency matrix $A(G)$ is defined to be the $n \times n$ $(0, 1)$ matrix (a_{ij}), where $a_{ij} = 1$ if and only if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. Then $A(G)$ is a symmetric $(0, 1)$ matrix with each diagonal entry equal to 0. The determinant $\det(\lambda I - A(G))$ is called the characteristic polynomial of G. The n eigenvalues of $A(G)$ are known as the n eigenvalues of the graph G.

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For any positive integer k, we denote by $P_k(\lambda)$ the characteristic polynomial of the adjacency matrix of the path of length $k-1$ on k vertices, i.e.,

$$P_k(\lambda) = \det(\lambda I - P_k), \quad k = 1, 2, 3, \ldots,$$

where $P_k = (p_{ij})$ is a symmetric $(0,1)$ matrix of order k, and $p_{ij} = 1$ iff $|i-j| = 1$, $1 \leqslant i, \ j \leqslant k$. Define $P_0(\lambda) = 1$. For an $n \times n$ matrix M, we also make the convention that $P_0(M) = I$, the identity matrix of order n. The matrices $P_k(M)$, $k = 0, 1, 2, \ldots,$ are called the path polynomials of the matrix M. Analogically, $P_k(A(G))$ are called path polynomials of graph G [1]. Recently, $P_k(\lambda)$ has been investigated by several authors, for example [1–4].

Definition 1 [1]. Let $A = A(G)$ be the adjacency matrix of G. If there exists a positive integer m, $m \geqslant 2$, such that

$$P_m(A) = \left[P_{m-2}(A) + I\right] + I, \qquad P_{m+1}(A) = P_{m-1}(A) + A,$$

then the least integer m is called the path-recursive period of G, denoted by $\mathrm{PRP}(G) = m$.

The following conjecture was posed by Shi [1].

Conjecture. There is a path-recursive period for any tree; indeed, for any connected graph.

In this paper, we consider the above conjecture. According to the discussions the answer to the conjecture is negative. Some results presented in [1,3] are included, too.

2. The path-polynomials

The following lemma is from [4].

Lemma 1. *Define $P_0(\lambda) = 1$, $P_1(\lambda) = \lambda$, where λ is any complex number. Then for $k \geqslant 2$ the path-polynomial $P_k(\lambda)$ is determined by*

(i) $P_k(\lambda) = \lambda P_{k-1}(\lambda) - P_{k-2}(\lambda)$,

(ii) $$P_k(\lambda) = \begin{cases} \dfrac{\sin(k+1)\theta}{\sin\theta}, & \theta = \arccos\frac{\lambda}{2}, \quad |\lambda| < 2, \\ k+1, & \lambda = 2, \\ (-1)^k(k+1), & \lambda = -2, \\ \dfrac{\sinh(k+1)\theta}{\sinh\theta}, & \theta = \operatorname{arccosh}\dfrac{\lambda}{2}, \quad |\lambda| > 2, \end{cases}$$

where $\sinh\theta$ *and* $\operatorname{arccosh}(\frac{1}{2}\lambda)$ *are the hyperbolic sine and anti-hyperbolic cosine.*

Proof. Since $P_k(\lambda) = \det(\lambda I - P_k)$, assertion (i) is obvious [5]. On the other hand, it is readily verified that (ii) is the unique solution to the difference equation (i). $\square$

We would like to point out that, when $|\lambda| < 2$, $P_k(\lambda)$ is the Chebyshev polynomial of the second kind.

Corollary 2. *For some number* λ_0, $P_k(\lambda_0) = 0$ *implies that* $|\lambda_0| < 2$.

The proof of Corollary 2 is by (ii) in Lemma 1.

The following result is from Theorem 1 of Shi [1] which is also a generalization of Theorem 2.5 in [3].

Corollary 3. *Let* A *be an* $n \times n$ *square matrix. If there exists an integer* r, $r \geqslant 1$, *such that* $P_r(A) = 0$, *then*
(a) $P_{t(r+1)-1}(A) = 0,\ t = 1, 2, 3, \ldots,$
(b) $P_{r+s}(A) = -P_{r-s}(A),\ 0 \leqslant s \leqslant r$, *and*
(c) $P_{2t(r+1)+s} = P_s(A),\ t = 0, 1, 2, \ldots,\ 0 \leqslant s \leqslant 2r + 1.$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the n eigenvalues of A. Then it is seen that $P_r(A) = 0$ implies that $P_r(\lambda_j) = 0$, $j = 1, 2, \ldots, n$. Therefore, from Corollary 2, we have that the spectral radius of A $\rho(A) = \max_{1 \leqslant i \leqslant n} |\lambda_i| < 2$. This also yields that

$$P_r(A) = \sin[(r + 1) \arccos(\tfrac{1}{2}A)] \sin \arccos(\tfrac{1}{2}A)^{-1} = 0,$$

which is equivalent to

(d) $\sin[(r + 1) \arccos(\frac{1}{2}A)] = 0,\ \rho(A) < 2.$

By (d) it is readily to show that assertions (a)–(c) are true. Here we give only the proof of (a), the others follow in a similar fashion. In fact, for any integer t, we have

$$\begin{aligned} P_{t(r+1)-1}(A) &= \sin[t(r + 1) - 1 + 1) \arccos(\tfrac{1}{2}A)] \sin \arccos(\tfrac{1}{2}A)^{-1} \\ &= \sin[t(r + 1) \arccos(\tfrac{1}{2}A)] \sin \arccos(\tfrac{1}{2}A)^{-1} \\ &= 0. \end{aligned}$$

This completes the proof of (a). $\square$

Corollary 3 is the main result of [1], see Theorem 1 in [1].

Remark 1. Lemma 2.4 given by Bapat and Lal [3] follows in a similar fashion. Furthermore, (1) and (2) in Lemma in [1] are equivalent (the proof (2) $\Longrightarrow$ (1) is by induction).

3. Disproof of the conjecture

Throughout this section, we always assume that $A = A(G)$ is the adjacency matrix of graph G. Thus A is a symmetric $(0, 1)$ matrix. Therefore, the n eigenvalues of A, say $\lambda_1, \lambda_2, \ldots, \lambda_n$, are real numbers. This yields the following.

Lemma 4. *For $m \geqslant 2$, the relations*

$$\begin{aligned} &P_m(A) = \left[P_{m-2}(A) + I\right] + I, \\ &P_{m+1}(A) = P_{m-1}(A) + A \end{aligned} \tag{1}$$

hold if and only if

$$\begin{aligned} &P_m(\lambda_i) = P_{m-2}(\lambda_i) + 2, \\ &P_{m+1}(\lambda_i) = P_{m-1}(\lambda_i) + \lambda_i, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{2}$$

Proof. Since A is real symmetric, the assertions are obvious. □

Theorem 5. *If $\lambda_i \neq 2$, $i = 1, 2, \ldots, n$, then G has a path-recursive period $\mathrm{PRP}(G) = m_0$ if and only if m_0 is the least number such that*

$$P_{m_0-2}(\lambda_i) = -1, \quad P_{m_0-1}(\lambda_i) = 0, \quad i = 1, 2, \ldots, n.$$

Proof. According to Lemma 4, if G has a path-recursive period $\mathrm{PRP}(G) = m_0$, then

$$\begin{aligned} &P_{m_0}(\lambda_i) = P_{m_0-2}(\lambda_i) + 2, \\ &P_{m_0+1}(\lambda_i) = P_{m_0-1}(\lambda_i) + \lambda_i, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{3}$$

Since we have, for an arbitrary m, that

$$P_{m+1}(\lambda_i) = \lambda_i P_m(\lambda_i) - P_{m-1}(\lambda_i), \quad i = 1, 2, \ldots, n,$$

then

$$\begin{aligned} &\lambda_i P_{m_0-2}(\lambda_i) + (2 - \lambda_i^2) P_{m_0-1}(\lambda_i) = -\lambda_i, \\ &2 P_{m_0-2}(\lambda_i) - \lambda_i P_{m_0-1}(\lambda_i) = -2. \end{aligned} \tag{4}$$

Now

$$\lambda_i \neq 2, \quad i = 1, 2, \ldots, n,$$

yields

$$P_{m_0-2}(\lambda_i) = -1, \quad P_{m_0-1}(\lambda_i) = 0, \quad i = 1, 2, \ldots, n. \tag{5}$$

Conversely, by using Lemma 4 and noting that (5) $\Longrightarrow$ (4) $\Longrightarrow$ (3) we can get the other part. □

Corollary 6. *If there exists an index i_0, $1 \leqslant i_0 \leqslant n$, such that $|\lambda_{i_0}| > 2$, then G has no path-recursive period.*

The proof of the claim can be verified by Lemmas 1,4 and Theorem 5.

Since there are a many graphs (including trees) whose spectral radii are larger than 2, by Corollary 6, this disproves the validity of the conjecture.

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