New Techniques for Bounding the Channel Capacity of Read/Write Isolated Memory*

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April 11, 2002

*This work was supported by Hong Kong RGC CERG grants HKUST6137/98E, HKUST6162/00E and DIMACS. *Email*: {xryong, golin}@cs.ust.hk

1 Introduction

- A serial, binary $(0, 1)$ memory is said to be *read isolated* if no consecutive positions can store 1's in the memory; it is said to be *write isolated* if rewritable and no two consecutive positions can be changed during rewriting.
- A *read/write isolated memory* is a binary rewritable storage medium obeying the two *different* restrictions.

2 2-D Constrained Codes for recording

- A *(d,k)-Runlength Limited (RLL)* constrained code is the code over binary alphabet $\{0, 1\}$ where the number d (k) is the minimum (maximum) permitted number of 0's separating consecutive 1's in each codeword.

  Examples: The $(1, 3)$-RLL constraint used in magnetic disc drivers, $(2, 10)$-RLL in compact audio discs, $(1, \infty)$-RLL is read isolated constraint. ...

- A 2-D code is a set of 2-D arrays or matrices that have constraints both horizontally and vertically. The work on 2-D codes is in its infancy but very active.

3 An Example

The read/write isolated constrained code is a 2-D code.

If we read vertically, then only the first matrix (codeword) of the following three matrices is fine:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \mathit{1} & \mathit{0} & 0 & 1 \\ \mathit{0} & \mathit{1} & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \mathit{1} & 1 & 0 & 1 \\ \mathit{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4 Transfer Matrices

Consider the set, V_m, of all possible '*strings*' that are allowed to appear as rows (or columns) in the $n \times m$ binary $(0, 1)$ arrays in a 2-D code S. We say that the pair (v_i, v_j) is *valid* if $v_i, v_j \in V_m$ and $\begin{pmatrix} v_i \\ v_j \end{pmatrix}$ (or (v_i, v_j)) does not violate the constraint.

The *transfer matrix* T_m is defined as $(T_m)_{ij} = 1$ if (v_i, v_j) is *valid* and 0 otherwise.

5 Examples on Transfer Matrices

The read/write isolated constraint has two *completely different* transfer matrices which generate according to the constraints in the vertical and the horizontal direction.

All length 2 strings of read isolated constraint (vertical) are

$$v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then the transfer matrix in this direction is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

6 Examples-Cont'd

All length 2 strings in horizontal direction are

$$\begin{aligned} v_1 &= 00 \\ v_2 &= 01 \\ v_3 &= 10 \\ v_4 &= 11. \end{aligned}$$

then the transfer matrix in this direction is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

7 Vertical, and Horizontal Transfer Matrices

1. Vertical transfer matrices A_k (Cohn, 1995):

   $$A_0 = 1, \quad A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A_k = \begin{pmatrix} A_{k-1} & \hat{A}_{k-2} \\ \hat{A}_{k-2}^t & A_{k-2} \end{pmatrix},$$

   where A_k is an $f_{k+2} \times f_{k+2}$ matrix, and $\hat{A}_{k-2} = \begin{pmatrix} A_{k-2} \\ 0 \end{pmatrix}$ is of $f_{k+1} \times f_k$, and f_k is the Fibonacci number.

2. Horizontal transfer matrices $\bar{A}_r$ (Golin, Yong, Zhang, Sheng; 2000):

   $$\bar{A}_r = \begin{pmatrix} \bar{A}_{r-1} & B_{r-1} \\ B_{r-1}^t & 0 \end{pmatrix}, \qquad B_{r-1} = \begin{pmatrix} \bar{A}_{r-2} & B_{r-2} \\ 0 & 0 \end{pmatrix}$$

   with initial conditions: $\bar{A}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$,

where $\bar{A}_r$ is of size $2^r \times 2^r$.

Note that they are symmetric. BUT totally different!

8 Channel Capacity

Let k be the size of the memory in binary symbols, r the lifetime of the memory in rewrite cycles and $N(k, r)$ the number of distinct sequences of r characters (each has length k).

- For fixed r, the *channel capacity*, measured in bits per symbol [12] is

$$C_r = \lim_{k \to \infty} \frac{1}{k} \log_2 N(k, r).$$

- For fixed k, the *channel capacity*, measured in bits per rewrite, is defined to be [15]

$$\bar{C}_k = \lim_{r \to \infty} \frac{1}{r} \log_2 N(k, r).$$

The *channel capacity* (of the read/write isolated memory), in bits per symbol per rewrite, is:

$$C = \lim_{k\to\infty} \frac{1}{k} C_k = \lim_{r\to\infty} \frac{1}{r} \bar{C}_r.$$

Let λ_k and μ_r be the largest eigenvalues of A_k and $\bar{A}_r$, respectively. Then $C = \lim_{k\to\infty} \frac{\log_2 \lambda_k}{k} = \lim_{r\to\infty} \frac{\log_2 \mu_r}{r}$, and for $\forall k, r \geq 1$,

$$max\{\log_2 \frac{\lambda_{2k}}{\lambda_{2k-1}}, \log_2 \frac{\mu_{2r}}{\mu_{2r-1}}\} \leq C \leq min\{\frac{\log_2 \lambda_k}{k}, \frac{\log_2 \mu_r}{r}\}.$$

9 The Formula of the Capacity

Let $N(m,n)$ be number of $m \times n$ arrays that satisfy constraints, and let A_m, $\bar{A}_n$ be the vertical and horizontal transfer matrices.

$$N(m,n) = \mathit{1}^t A_m^{n-1} \mathit{1} = \mathit{1}^t \bar{A}_n^{m-1} \mathit{1},$$

$$\begin{aligned} C &= \lim_{n,m \to \infty} \frac{1}{nm} \log_2 N(m,n;S) \\ &= \lim_{n,m \to \infty} \frac{1}{nm} \log_2(\mathit{1}^t A_m^{n-1} \mathit{1}). \end{aligned}$$

The capacity is the amount of information (that is stored or transmitted) per bit.

10 Combinatorial Interpretation

Generally, these problems can be interpreted, combinatorically, as

- Count the number, say $N(k, r)$, of $k \times r$ matrices that satisfy a given constraint (horizontally and vertically ...).
- Analyze the asymptotic properties of the number $N(k, r)$ for k, r tend to infinity.

11 History and Results

- The *read isolated* constraint occurs in magnetic recording and optical recording. Freiman and Wyner ([7], 1964) first considered this problem and showed that the capacity of such a memory is $\log_2 \varphi = 0.694...$
- The *write isolated* restrictions arises in rewritable optical discs [5]. Again, the capacity is $\log_2 \varphi$ (Cohen and Zemor, 1993).

12 History and Results—Cont'd

- Cohn (1995) did the first analysis of the capacity C of read/write isolated constraint and proved

  $$0.509... \le C \le 0.560297....$$

  These bounds were later improved to (Golin et al, 2000)

  $$0.53500... \le C \le 0.55209....$$

- This paper provides the following

  $$0.5350150... \le C \le 0.5396225...$$

  by introducing the *compressed matrix technique*.

13 Upperbounding on the Capacity

Let $Q_0(x) = 1, Q_1(x) = \begin{pmatrix} 1 & x \\ x & x \end{pmatrix}$, and

$$Q_m(x) = \begin{pmatrix} Q_{m-1}(x) & \hat{Q}_{m-2}(x) \\ \hat{Q}_{m-2}(x)^t & Q_{m-2}(x) \end{pmatrix}, \ \hat{Q}_{m-2}(x) = \begin{pmatrix} Q_{m-2}(x) \\ 0 \end{pmatrix},$$

where Q_m is of size $f_{m+2} \times f_{m+2}$. Then

Lemma 1. Let $\rho(Q_m(x))$ be the largest eigenvalue of $Q_m(x)$. Then,

$$\forall m \geq 1, \quad C \leq \frac{\log_2 \rho\left(Q_m\left(2^{-C}\right)\right)}{m}.$$

Theorem 1. For $\forall x \in [0.69015866, 1)$ and $\forall m \geq 1$, the largest eigenvalue of compressed matrix $Q_m(x)$ satisfies

$$2^C \leq (\rho(Q_m(x)))^{\frac{1}{m}} < (\rho(A_m))^{\frac{1}{m}} = \lambda_m^{1/m}.$$

Lemma 2. Given an $n \times n$ matrix

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1k} \\ T_{21} & T_{22} & \cdots & T_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ T_{k1} & T_{k2} & \cdots & T_{kk} \end{pmatrix} = (T_{ij}),$$

where T_{ij} is an $n_i \times n_j$ matrix, $\Sigma_{i=1}^{k} n_i = n$, define

$$||T_{ij}||_2 = \sup_{x_j \neq 0} \frac{||T_{ij}x_j||_2}{||x_j||_2}, \ i, \ j = 1, \ 2, \ \cdots, \ k,$$

and let $M = (||T_{ij}||_2)$. Then

$$\rho(T) \leq \rho(M),$$

where $\rho(X)$ is the spectral radius of matrix X. The matrix M is called a *compressed matrix* of T.

15 The Idea of Their Proofs—Cont'd

Using the recurrence relations for the vertical transfer matrices A_k:

$$A_k = \begin{pmatrix} A_{k-1} & \hat{A}_{k-2} \\ \hat{A}^t_{k-2} & A_{k-2} \end{pmatrix}, A_k = \begin{pmatrix} A_{k-2} & \hat{A}_{k-3} & A_{k-2} \\ \hat{A}^t_{k-3} & A_{k-3} & 0 \\ A_{k-2} & 0 & A_{k-2} \end{pmatrix}, \ldots,$$

we introduce the compressed matrices

$$M_1 = \begin{pmatrix} ||A_{k-1}||_2 & ||\hat{A}_{k-2}||_2 \\ ||\hat{A}^t_{k-2}||_2 & ||A_{k-2}||_2 \end{pmatrix} = \begin{pmatrix} \lambda_{k-1} & \lambda_{k-2} \\ \lambda_{k-2} & \lambda_{k-2} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} ||A_{k-2}||_2 & ||\hat{A}_{k-3}||_2 & ||\hat{A}_{k-2}||_2 \\ ||\hat{A}^t_{k-3}||_2 & ||A_{k-3}||_2 & 0 \\ ||A_{k-2}||_2 & 0 & ||A_{k-2}||_2 \end{pmatrix} = \begin{pmatrix} \lambda_{k-2} & \lambda_{k-3} & \lambda_{k-2} \\ \lambda_{k-3} & \lambda_{k-3} & 0 \\ \lambda_{k-2} & 0 & \lambda_{k-2} \end{pmatrix}, \ldots,$$

By considering the matrices $\{M_r/\lambda_{k-r}\}$, and then using Lemma 2, we obtained the matrices $\{Q_m(x)\}$.

m	F_{m+2}	$\rho(Q_m(x_0))$	$\frac{\log_2(\rho(Q_m(x_0)))}{m}$
1	2	1.5524196988	0.6345186448
2	3	2.1732673267	0.5599328234
3	5	3.2543210622	0.5674521968
4	8	4.6152889202	0.5516052411
5	13	6.8116188717	0.5535995429
6	21	9.7408000844	0.5473400461
7	34	14.2668740681	0.5477996234
8	55	20.5065208518	0.5447513549
9	89	29.9058714421	0.5447063169
10	144	43.0634915800	0.5428393393
11	233	62.7213694907	0.5428077414
12	377	90.6138568737	0.5418049818
13	610	131.2975410982	0.5412843145
14	987	190.3410264771	0.5408887676
15	1597	276.1769813000	0.5406299513
16	2584	399.7392197852	0.5401822073
17	3181	579.6907368666	0.5399493895
18	5765	839.3943441870	0.5396224962

Table 1: F_m is the mth Fibonacci number. $\rho(Q_m(x_0))$ is the largest eigenvalue of the $F_{m+2} \times F_{m+2}$ compressed matrix $Q_m(x_0)$.

Table 1 gives numerical results for fixed values $x_0 = 0.69015866, m = 18$. The $m = 18$ entry of Table 1 yields our new upper bound of $C < 0.53962245$.

16 Lowerbounding on the channel capacity C

Lemma 3. Let $p(x) = det(xI - \bar{A}_r)$ be the characteristic polynomial of the $2^r \times 2^r$ horizontal transfer matrix $\bar{A}_r$. Then

$$p(x) = det(xI - \bar{A}_r) = x^{2^{r-1}} det(xI - M_r),$$

where M_r is the $2^{r-1} \times 2^{r-1}$ nonnegative matrix given by

$$M_r = \begin{pmatrix} \bar{A}_{r-4} & B_{r-4} & \bar{A}_{r-4} & B_{r-4} & \bar{A}_{r-4} & 2\bar{A}_{r-4} & 3\bar{A}_{r-4} & 6B_{r-4} \\ B^t_{r-4} & 0 & 0 & 0 & B^t_{r-4} & 2B^t_{r-4} & 0 & 0 \\ \bar{A}_{r-4} & 0 & 0 & 0 & \bar{A}_{r-4} & 0 & 0 & 0 \\ B^t_{r-4} & 0 & 0 & 0 & B^t_{r-4} & 0 & 0 & 0 \\ \bar{A}_{r-4} & B_{r-4} & \bar{A}_{r-4} & B_{r-4} & 0 & 0 & 0 & 0 \\ \bar{A}_{r-4} & B_{r-4} & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{A}_{r-4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B^t_{r-4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Table 2 gives the numerical results of our computations on the largest eigenvalues of $\bar{A}_r$ and the bounds of the capacity. We computed the largest eigenvalues of the compressed matrix DM_rD^{-1} using the *Power Method*. The first 8 rows of Table 2 are already appeared in [9]; the remaining results were calcu-

r	2^r	μ_r	$\log_2 \frac{\mu_r}{\mu_{r-1}}$
1	2	1.618	
2	4	2.302775637	0.5091622465
3	8	3.346462191	0.5392628601
4	16	4.845619214	0.5340443229
5	32	7.021562462	0.5351110622
6	64	10.17359346	0.5349653450
7	128	14.74105370	0.5350103028
8	256	21.35908135	0.5350099454
9	512	30.948359597568	0.535010307223
10	1024	44.842824649376	0.535014214230
11	2048	64.975322373528	0.535014730251
12	4096	94.146459043118	0.535014947823
13	8192	136.414197132806	0.535015053352
14	16384	197.658348650048	0.535015093868

Table 2: μ_r is the largest eigenvalue of the $2^r \times 2^r$ transfer matrix $\bar{A}_r$. For $r > 8$ the μ_r were calculated by calculating the largest eigenvalues of DM_rD^{-1}.

lated with the new technique described in this section.

Plugging the values in Table 2 into the inequalities on page 5 derives our new lower bound $0.5350150 \leq C$.

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