Algebraic and Combinatorial Properties of the Transfer Matrix of the 2-Dimensional $(1,\infty)$-Runlength Limited constraint*

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June 5, 2002

*This work was supported by Hong Kong CERG grant 652/95E, 6137/98E, 6162/00E, HKAoE/E-01/99, and DIMACS. *Email*: { xryong, golin }@cs.ust.hk.

1 Introduction

The codes here are the ones used in magnetic, digital or optical recordings.

- A *(d,k)-Runlength Limited (RLL)* constrained code is the set of codewords over binary alphabet $\{0, 1\}$ all satisfying the constraints that the number d (k) is the minimum (maximum) permitted number of 0's separating consecutive 1's in a legal binary sequence. For example,

  $$1001000100000100000000001000$$

  is a word that satisfies the *(2,10)-RLL* constraint used in compact audio discs.

- A 2-D code has constraints both horizontally and vertically. The two constraints may be different.

  The read/write isolated constraint is an example. (constraints in reading no two consecutive 1's and in each rewriting cycle, no two consecutive positions are allowed to change.)

2 2-D *(d,k)-RLL* constraint

The 2-D (d, k)-RLL code $S_{d,k}^{(2)}$ satisfies the RLL constraint both horizontally and vertically.

For example (if we read in y direction and write in x direction):

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{1} & 0 & 0 & 1 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first matrix is fine, but the second is not permitted.

3 The Capacity

- *The capacity* $cap(S)$ measures the growth rate of the number $N(m, n; S)$ of $m \times n$ arrays in S, *i.e.*, the number of $m \times n$ $(0, 1)$-matrices that satisfy the constraints in two directions,

$$cap(S) = \lim_{n,m \to \infty} \frac{\log_2 N(m, n; S)}{nm},$$

intuitively, it represents the maximum amount of information that is transmitted or stored per bit in communication.

4 The history

The 2-D *RLL* Constrained code $S_{1,\infty}^{(2)}$ also arises in statistical physics, graph theory of coding theory, until recently it was attached separatingly by different fields.

- also called *hard-square*, or *hard-core lattice gas* system (Burton and Steif, 1994)
- Engel (1982) called it *Fibonacci number of a lattice*,
- Calkin and Wilf (1998) called it *independent sets in grid graph.* Denote it simply by S_{hs}.
- Combinatorically, it contains all matrices of size $m \times n$ that do not have adjacent horizontal or vertical '1's.

5 Previous results

As far as we know in estimating the Shannon capacity, there has been no one who has paid much attention to the algebraic and combinatorial properties of the transfer matrix for a constrained code.

For the Hard Square code, Weber seems to be the first to consider its capacity. Weber (1988, [6]) obtained

$$.53602 \leq cap(S_{hs}) \leq .63598,$$

and then Engel (1990, [2])

$$.58789 \leq cap(S_{hs}) \leq .59756,$$

and then Calkin and Wilf (1998,[5]) proved that

$$.587891 \leq cap(S_{hs}) \leq .588339.$$

Now they have been further improved by Nagy and Zeger [14] to be

$$.587891161775 \leq cap(S_{hs}) \leq .587891161868.$$

6 The objective of this talk

Focuses on deriving the properties of transfer matrix of the *Hard Square* system S_{hs}. Similar properties of the other constrained codes can be obtained using the techniques and ideas.

- We derive some algebraic and combinatorial properties of the transfer matrix of the *Hard Square* system S_{hs}.

7 Transfer matrix

Let C_m $(m \geq 0)$ be the set of all column$(m+1)$-vectors v_i of $0^{'}s$ and $1^{'}s$, such that v contains no two consecutive $1^{'}s$.

Let $T_{F_{m+3}} = (t_{ij})$ where $t_{ij} \stackrel{\text{def}}{=} \widehat{v_i v_j}$ is 1 if the concatenation $v_i v_j$ satisfies the constraints and is 0 otherwise. Then $T_{F_{m+3}}$ is called the **transfer matrix** of the problem.

The number of $m \times n$ matrices $f(m,n)$ that satisfy the constriants is given by

$$f(m,n) = \mathit{1}^t \cdot T^n_{F_{m+3}} \cdot \mathit{1},$$

where $\mathit{1}$ is the vector of all$1^{'}s$.

8 The capacity and the transfer matrix

$$
\begin{aligned}
cap(S_{hs}) &= \lim_{n,m\to\infty} \frac{\log_2 f(m,n)}{nm} \\
&= \lim_{n,m\to\infty} \frac{\log_2 \mathit{1}^t \cdot T^n_{F_{m+3}} \cdot \mathit{1}}{nm} \\
&= \lim_{m\to\infty} \log_2 {\lambda_m}^{1/m} \\
&= \lim_{m\to\infty} \log_2 \frac{\lambda_{m+1}}{\lambda_m},
\end{aligned}
$$

and

$$cap(S_{hs}) \geq \log_2 \frac{\lambda_{2m+1}}{\lambda_{2m}},$$

where λ_m is the largest eigenvalue of $T_{F_{m+3}}$.

9 Characterizations of transfer matrix

We can easily find its transfer matrices.

Lemma 1. If we arrange the $(m+1)$(0,1)-sequences in lexicographcal order, then the transfer matrix is given by

$$T_{F_{m+3}} = \begin{pmatrix} T_{F_{m+2}} & T_{F_{m+2}}^{(F_{m+1})} \\ * & 0_{F_{m+1}} \end{pmatrix},$$

where $T_{F_{m+2}}^{(F_{m+1})}$ is the first F_{m+1} columns of $T_{F_{m+2}}$, i.e., $T_{F_{m+2}} = (T_{F_{m+2}}^{(F_{m+1})}, \cdot)$, $0_{F_{m+1}}$ is the $F_{m+1} \times F_{m+1}$ null matrix, the "$*$" signifies the transpose part.

The first three matrices:

$$T_{F_3} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, T_{F_4} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, T_{F_5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

10 The L^tDL decomposition

Lemma 3. The transfer matrix $T_{F_{m+3}}$ has the following L^tDL decomposition:

$$T_{F_{m+3}} = L^t_{F_{m+3}} D_{F_{m+3}} L_{F_{m+3}},$$

where

$$L_{F_{m+3}} = \begin{pmatrix} L_{F_{m+2}} & \begin{pmatrix} L_{F_{m+1}} \\ 0 \end{pmatrix} \\ 0 & L_{F_{m+1}} \end{pmatrix}, L_{F_3} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, L_{F_4} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D_{F_{m+3}} = \begin{pmatrix} D_{F_{m+2}} & 0 \\ 0 & -D_{F_{m+1}} \end{pmatrix}, D_{F_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, D_{F_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

11 The distributions of its eigenvalues

Theorem 1. Let P_m and N_m be the numbers of positive and negative eigenvalues of $T_{F_{m+3}}$. Then

$$P_m - N_m = -\frac{2}{\sqrt{3}} sin \frac{m\pi}{3},$$

where $N_m = N_{m-1} - N_{m-2} + F_{m-2}$ with $N_0 = 1$ and $N_1 = 2$.

12 The inverse of the matrix

Corollary 1. The inverse of $T_{F_{m+3}}$ is an $(-1,\ 0,\ 1)-$matrix (the elements are $-1, 0, 1$) , and is given recursively by

$$T^{-1}_{F_{m+3}} = \begin{pmatrix} \begin{pmatrix} -T^{-1}_{F_m} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} T^{-1}_{F_m} \\ 0 \end{pmatrix} & T^{-1}_{F_{m+1}} \\ (T^{-1}_{F_m}, 0) & -T^{-1}_{F_m} & 0 \\ T^{-1}_{F_{m+1}} & 0 & -T^{-1}_{F_{m+1}} \end{pmatrix}.$$

13 The power of the matrix

Lemma 4. The transfer matrix $T_{F_{m+3}}$ satisfies the following relations: $T^k_{F_{m+3}} P_{F_{m+3}} = P_{F_{m+3}} T^k_{F_{m+3}}, k = \pm 1, \pm 2, \cdots$, where $P_{F_{m+3}}$ is a symmetric permutation matrix of size F_{m+3}.

This gives $T^2_{F_{m+3}} = A^2_{F_{m+3}}$, where $A_{F_{m+3}} = T_{F_{m+3}} P_{F_{m+3}} = P_{F_{m+3}} T_{F_{m+3}}$.

Corollary 3. If λ_i and μ_i are the eigenvalues of transfer matrix $T_{F_{m+3}}$ and matrix $A_{F_{m+3}}$, respectively, then we have

$|\lambda_i| = |\mu_i|$ $(i = 1, 2, \cdots, F_{m+3})$. So $T_{F_{m+3}}$ and $A_{F_{m+3}}$ share the same largest eigenvalue. Furthermore,

$$f(m, n) = \mathit{1}^t T^n_{F_{m+3}} 1 = \mathit{1}^t A^n_{F_{m+3}} 1.$$

14 The general properties

Similar relations hold for other constraints. Even if the corresponding transfer matrix, say T, is not symmetric, then we have the per-symmetric property:

$$T^k P = P(T^t)^k, k = 0, 1, 2, \cdots.$$

where the P is a symmetric permutation matrix. Note that the graph of P is of loops plus cycles of order 2.

15 The order of the recursive relation that $f(m,n)$ satisfies

$f(m,n)$ satisfies a recursive relation with order

$$\begin{cases} (F_{m+3} + F_{\frac{m+3}{2}})/2, & if\ m\ is\ odd; \\ (F_{m+3} + F_{\frac{m+6}{2}})/2, & if\ m\ is\ even. \end{cases}$$

Example 3. Following are the first three recursive relations for $f(n,m) = \mathit{1}^t T^m_{F_{n+3}} \mathit{1}$ (for $m = 1, 2, 3$). Their recurrence relations have orders $2, 4, 5$, respectively.

$$\begin{aligned} f(n,1) &= 2f(n-1,1) + f(n-2,1); \\ f(n,2) &= 2f(n-1,2) + 6f(n-2,2) - f(n-4,2); \\ f(n,3) &= 4f(n-1,3) + 9f(n-2,3) - 5f(n-3,3) \\ &\quad -4f(n-4,3) + f(n-5,3). \end{aligned}$$

Their initial conditions are given by $f(m,n) = f(n,m) = \mathit{1}^t T^n_{F_{m+3}} \mathit{1}$, for example, $f(1,1) = 7, f(2,1) = 17$, and so on.

16 The analytic expression of $f(m,n)$

Theorem 3. Let $\varphi(\lambda) = det(\lambda I - T)$ be the characteristic polynomial of T. Then

$$\begin{aligned} f(m,n) &= \frac{|\lambda_m I - T_{11}|}{\varphi'(\lambda_m)}\lambda_m^{n+2} + \frac{|\lambda_{j_2} I - T_{11}|}{\varphi'(\lambda_{j_2})}\lambda_{j_2}^{n+2} + \cdots \\ &+ \frac{|\lambda_{j_{r_m}} I - T_{11}|}{\varphi'(\lambda_{j_{r_m}})}\lambda_{j_{r_m}}^{n+2}, \end{aligned}$$

where T_{11} is the bottom-right $(F_{m+3} - 1) \times (F_{m+3} - 1)$ principal submatrix of T, *i.e.*, $T = \begin{pmatrix} 1 & \mathit{1}^t \\ \mathit{1} & T_{11} \end{pmatrix}$, and all coeffients $\frac{|\lambda_j I - T_{11}|}{\varphi'(\lambda_j)} > 0$.

17 Numerical computations

m	$\frac{\lambda_{m+1}}{\lambda_m}$
1	1.4920660376475357643782160628584
2	1.5041673682066925408320381142165
3	1.5029282260930080313112631785226
4	1.5030600955153464778914585820045
5	1.5030467676434921566255567236369
6	1.5030482087273507836965767785786
7	1.5030480675735632786709176203759
8	1.5030480837106775097713484237842
9	1.5030480822893138617353323026629
10	1.5030480824838507827311370714394
11	1.5030480824723636135517003809521
12	1.5030480824752323465499639032462
13	1.5030480824752615741231348797605

From the table,

$$
\begin{aligned}
cap(S_{hs}) &> \log_2 1.5030480824752323465499639032462 \\
&= 0.587891161775232....
\end{aligned}
$$

We computed them using the recursive relations in Lemma 1 and the *Power Method*. The corresponding two matrices are of sizes $377, 610$, respectively.

18 Conclusion and Conjecture

We considered some algebraic and combinatorial properties of the transfer matrix of the *Hard Square* system. Some similar results can be obtained by making use of the approaches.

We would like to pose the following Conjecture. It seems true for the transfer matrices of the *Read/Write Isolated* memory ([23], [24]) and the *Hard Square* system.

Conjecture. If the transfer matrix T of a constrained system is symmetric, and if λ is an eigenvalue of T, and λ is not 0 or -1, then the number $-\frac{1}{\lambda}$ is also an eigenvalue of T.

19 The case of Hard Square system

In the case of the *Hard Square* system, T is invertible. If the conjecture is true, then we can prove easily that

$$\frac{\mathit{1}^{\,t}T^{-2n}\mathit{1}}{\mathit{1}^{\,t}T^{-(2n-2)}\mathit{1}} \leq \frac{\mathit{1}^{\,t}T^{-(2n+2)}\mathit{1}}{\mathit{1}^{\,t}T^{-2n}\mathit{1}} \leq -\lambda_m.$$

where λ_m is the largest eigenvalue of T. This would give a very good upper bound of its capacity.

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